Problem Set 7 Solutions

Problem 1. **[10 points]**

Determine whether each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, is an injection, surjection or a bijection. Also, give the inverse if it exists.

(*Note: The inverse of a function is defined in Rosen page 64.*)

- $f(x) = e^x$
- $f(x) = x^2 + 3$
- $f(x) = (x+1)/(x^2+2)$
- $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$

- $f(x) = e^x$ is an injection because $x \neq y \Rightarrow e^x \neq e^y$. It is not a surjection because it maps $\mathbb{R}$ onto $\mathbb{R}^+$. Hence, it is not a bijection, and so it does not have an inverse.
- $f(x) = x^2 + 3$ is not a surjection because it maps $\mathbb{R}$ onto $[3, \infty)$. It's not an injection, because -1 and $+1$ have the same image. Hence, it's not a bijection and it does not have an inverse.
- $f(x) = (x+1)/(x^2+2)$ is not an injection because 0 and 2 have the same image. It's not a surjection because $+1$ doesn't have a pre-image (equivalently the equation $x^2 - x + 1$ has no real solutions). Therefore it's not a bijection and it doesn't have an inverse.
- This function is a surjection, because for every $x \in \mathbb{R}, x \neq 0$, there exists a $y = 1/x$ such that $f(y) = x$. 0 also has a pre-image, namely 0. It's also a injection because $\frac{1}{x} = \frac{1}{y} \Rightarrow x = y, \forall x, y \neq 0$. Moreover, if $x \neq 0$, then it is mapped on a non-zero real number (namely $1/x$). Therefore f is a bijection. It's inverse is itself.

Problem 2. **[10 points]**

Prove that the composition of two injective functions is injective.

Let $f : B \to C$ and $g : A \to B$, so that $f \circ g : A \longrightarrow C$, for sets $A, B,$ and C. Let x and y be distinct elements of A. Since g is one-to-one, $g(x)$ and $g(y)$ are distinct elements of B. Since f is one-to-one, $f(g(x)) = f \circ g(x)$ and $f(g(y)) = f \circ g(y)$ are distinct elements of C. Hence, $f \circ g$ is one-to-one.

Problem 3. **[10 points]**

Let A and B be finite sets.

(a) Prove that if $|A| \geq |B|$, then there exists a surjection from A to B.

Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$. Create a function f by setting $f(a_i) = b_i$ for $i \leq |B|$ and $f(a_i) = b_{|B|}, \forall i > |B|$ (if any). This function is obviously a surjection.

(b) Prove that if $|A| \leq |B|$, then there exists an injection from A to B.

Let $f : A \to B$ be the function defined by $f(a_i) = b_i$. f is an injection.

Problem 4. **[20 points]**

(a) **[10 points]** If $\mathcal{P}$ is any set of $n+1$ integers selected from $1, 2, 3, \ldots, 2n$, prove that there are two relatively prime integers in $\mathcal{P}$.

The set $\mathcal{P}$ must contain one of the n pairs of consecutive integers: $1, 2$ or $3, 4$ or $5, 6$ or $\ldots$ or $2n-1, 2n$, which are clearly relatively prime. Let the set of $n+1$ integers represent the $n+1$ pigeons, and n pairs of consecutive integers that are relatively prime, represent the holes. Then by pigeonhole principle, there must exist at least one pair of consecutive integers among $n+1$ integers in $\mathcal{P}$.

(b) **[10 points]** Consider a sequence of k integers: $a_1, a_2, \ldots, a_k$. Show that there is some subset of these terms whose sum is divisible by k.

First look at the k sums $a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + \ldots + a_k$. If one of these is divisible by k, we are done. If not, each of these has a nonzero remainder when divided by k. Since the only possible nonzero remainders are $1, 2, 3, \ldots, k-1$, by pigeonhole principle two of these sums must have the same remainder r when divided by k. The difference of two such integers is: $(qk+r)-(pk+r) = (q-p)k$, for integers p, q, which is divisible by k. In that case, if $a_1 + a_2 + \ldots + a_n$ and $a_1 + a_2 + \ldots + a_m$ are the sums in the question and $n > m$, then $a_{m+1} + a_{m+2} + \ldots + a_n$ is the required sum that is divisible by k.

Problem 5. **[10 points]**

Suppose that you have a scale on which you can compare any two groups of balls and produce three outcomes (lighter, heavier, or equal). You are given five balls. All but one have the same weight. You are supposed to find which ball differs and how (lighter or heavier). Can you do this with two weighings?

There are three possible outcomes for each weighting (heavier, lighter, or equal) so there are $3^2 = 9$ possible outcomes of the experiment There are five ways to pick the ball of different weight and two ways in which the weight can differ (lighter or heavier). So there are ten different case and only nine possible outcomes. Therefore the problem cannot be solved with two weighings.

Problem 6. [**10 points**] Let $\mathcal{S}_i, 1 \leq i \leq \infty$ be a family of countable sets. Prove that $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is countable.

Proof. Since each S_i is countable, it's elements can be listed in a sequence. Let s_{ik} be the kth element in the listing of S_i. We list the sets in the following array

$$\begin{array}{ccccc} s_{11} & s_{12} & s_{13} & s_{14} & \cdots \\ s_{21} & s_{22} & s_{23} & s_{24} & \cdots \\ s_{31} & s_{32} & s_{33} & s_{34} & \cdots \\ s_{41} & s_{42} & s_{43} & s_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We can count the elements using the dove-tailed interleaving shown in class, to get the listing for their union. The resulting list is $\{s_{11}, s_{12}, s_{21}, s_{13}, s_{22}, s_{31}, \ldots\}$.

Since we have been able to give a valid listing, $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is countable. ∎

Problem 7. [**10 points**]

Use a diagonalization argument to show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Proof. Assume, for the purposes of contradiction, that this set is countable. Then there is a listing of the function in the set, $f_1, f_2, \ldots$. Now define f such that $f(i) = (f_i(i) + 5)$ mod 10. Then f is not in the list, because f differs from each f_i in the ith value. We have established a contradiction. We conclude that the set is uncountable. ∎

Problem 8. [**10 points**]

Let F be the set of all functions $f : \mathbb{R} \to \{0, 1\}$. Show that there is a bijection from $2^{\mathbb{R}}$ to F.

Let ϕ: $2^{\mathbb{R}} \to F$ be the function mapping any set $A \subseteq \mathbb{R}$ to the function ϕ_A defined as follows:

$$\phi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

We prove that ϕ is a bijection, i.e., it is injective and surjective.

- Injective: Let A and B be two subsets of $\mathbb{R}$ such that $A \neq B$. A and B cannot be both empty, so assume without loss of generality that there exists some x such that $x \in A$ but $x \notin B$. We have $\phi_A(x) = 1 \neq 0 = \phi_B(x)$. Therefore $\phi_A \neq \phi_B$ and ϕ is injective.
- Surjective: Let f be a function from $\mathbb{R}$ to $\{0, 1\}$. We want to show that $f = \phi_A$ for some A. Define the set $A = \{x \in \mathbb{R} \mid f(x) = 1\}$. We have $\phi_A(x) = f(x)$ for all $x \in \mathbb{R}$. Therefore $\phi_A = f$ and ϕ is surjective.

Problem 9. **[10 points]**

A real number is *algebraic* if it is the root of a polynomial with rational number coefficients. Algebraic numbers form an important part in the study of (what else) modern algebra. A number that is not algebraic is called *transcendental.*

(a) Prove that for any particular d, the set of degree d polynomials with rational coefficients is countable.

Let P_d be the set of degree d polynomials with rational coefficients. The function

$$f\colon (q_0, q_1, \ldots, q_d) \mapsto q_0 x^d + q_1 x^{d-1} + \cdots + q_d$$

is a surjection from $(\mathbb{Q} - \{0\}) \times \mathbb{Q}^d$ to P_d. Since the set $(\mathbb{Q} - \{0\}) \times \mathbb{Q}^d$ is countable (the Cartesian product of finitely many countable sets is countable), then also the set P_d is countable.

(b) Prove that the set of all roots of degree d polynomials with rational coefficients is countable (you may use the fact that a degree d polynomial has at most d roots).

Since each degree d polynomial has at most d roots we can give a surjection from $P_d \times \{1, \ldots, d\}$ to the set of all roots of degree d polynomials with rational coefficients. For example we can define $f(p(x), i) = x_i$, where $x_1, \ldots, x_d$ are the roots of $p(x)$ (if $p(x)$ has less then d roots, just repeat the last one many times).

(c) Prove that the set of algebraic numbers (which is just the union of the sets in the previous part) is countable (*Hint: use the result of problem 6*).

The set of algebraic numbers is the union of the sets of the roots of polynomials with rational coefficients of degree d, for all d. This is a countable union of countable sets, and from the result in part (a) it is countable.

(d) Prove that there is a transcendental number.

If there were no transcendental numbers, all numbers would be algebraic, i.e., the set of algebraic numbers equals $\mathbb{R}$. But this is not possible because the reals are not countable.