Practice Quiz 1

This practice quiz contains problems from quizes of past semesters. There is some new material covered this semester that is not covered in this practice quiz. Also, expect the actual quiz to be much shorter (7 to 10 problems).

Problem 1. Use induction to prove that

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

whenever n is a natural number.

Problem 2. Why can't you find an integer x for which

$$384 \cdot x = 1 \pmod{566}?$$

Problem 3. Prove the set identity
$((A - B) \cup (B - A))^c = (A^c \cup B) \cap (A \cup B^c)$

Problem 4. Use the following tables to describe the properties of the various functions. Write yes or no in each box. Don't give proofs (it'd be too hard to squeeze them into the boxes). **Pay careful attention** to the specified domains and ranges.

- $f : \mathbb{N} \to \mathbb{N},\ f(x) = 5x.$
- $g : \mathbb{R}^+ \to \mathbb{N},\ g(x) = \lfloor x \rfloor.$
- $h : \mathbb{R} \to \mathbb{R}^+,\ h(x) = e^x.$

Give an inverse if one exists (under the given domain and range).

function	injective	surjective	inverse
$f(x) = 5x$			
$g(x) = \lfloor x \rfloor$			
$h(x) = e^x$			

Problem 5. Consider the following algorithm, which takes natural numbers a and b, and returns an ordered pair of natural numbers.

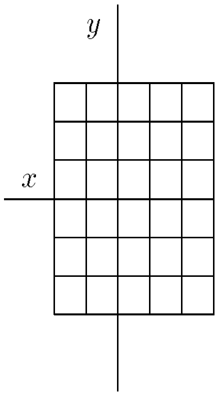
Algorithm Div(a, b)

if $(a < b)$
    return $(0, a)$
else
    $(q, r) \leftarrow$Div$(a - b, b)$
    return $(q + 1, r)$

(a) Prove Div terminates in at most a steps.

(b) Prove that Div returns the quotient and remainder of a on division by b.

Problem 6. David is located at position $(1, 0)$ on the two-dimensional $(\mathbb{N} \times \mathbb{N})$ x-y plane. His house is located at the origin. David wants to go home. But it is windy, and each time David takes a one-unit step east or west, he also gets blown a one-unit step north or south (he can pick which).



(a) On the grid above, mark with X's all the points that David can reach in one step. Mark with O's the points that David can reach in two steps. (The axes labeled x and y intersect at David's house.)

(b) Give an informal explanation for why David can never get home.

(c) Describe a state machine that represents David's motion, and list its transitions.

(d) Prove that David can never get home.

Problem 7. Consider the following algorithm that takes an array (list) $A[1 \ldots n]$ of numbers such that $A[1] < A[2] < \cdots < A[n]$, and a number q, and is supposed to find q if it is present in the list.

Find(A, n, q)

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i ← 1
j ← n + 1
While (j − i > 1)
    choose some k with i < k < j
    if A[k] ≤ q
        i ← k
    else
        j ← k
Return i
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(a) prove this function terminates

Now suppose that $q = A[r]$ for some r.

(b) State an invariant of the while loop relating i, j, and r. The invariant should be "obviously provable by induction" but you need not prove it.

(c) Assuming the invariant, prove that Find returns r.

Problem 8. An infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$ is defined in such a way that $a_1 = 14, a_2 = 35$, and each later a_i, $i \geq 3$, is obtained using one of the following rules:

Rule A Choose any two preceding elements of the sequence and define a_i to be their sum.

Rule B Choose any number $k \geq 1$ of preceding elements of the sequence,and define a_i to be their product divided by 7^{k-1}.

Use the well-ordering principle to prove that every element in the sequence is a multiple of 7. (*Hint: In case you have forgotten, the well-ordering principle says that every nonempty set of naturals has a smallest element.*)

Problem 9. Prove that among any set of 150 natural numbers, there must be three numbers, a, b, and c, such that all the pairwise differences, $a - b, a - c, b - c, b - a, c - a, c - b$, are multiples of 70.

Problem 10. For each of the following sets, either use diagonalization to prove that the set is uncountable, or else use dovetailing or another technique to show that the set is countable.
(If an answer to one part of the question follows from an answer you have already given to another part, then you need only indicate this – you don't need to give a direct proof of the part that follows.)

(a) The set E of all the even natural numbers.

(b) The set of all subsets of E.

(c) The set of all finite subsets of E.

(d) The set of all even-size finite subsets of E.

Problem 11. Give tight (Θ) bounds for

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

You do not need to prove your answer formally, but you should show how you derived it.

Problem 12. For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with O, Ω, Θ, o or ω, depending on whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$, $f(n) = o(g(n))$ or $f(n) = \omega(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write only the strongest one (a relation X is stronger than another relation Y if X implies Y but the converse does not hold). The first line is an example solution. Note: $\lg n = \log_2 n$.

$$2n = \underline{\qquad} (n)$$

$$\lg(n^3) = \underline{\qquad} (\lg n)$$

$$n! = \underline{\qquad} (4^n)$$

$$2^{\log_3 n} = \underline{\qquad} (n^2)$$

$$\sum_{i=1}^{n} 1/i = \underline{\qquad} (1)$$

$$n^{4.01} = \underline{\qquad} (n^4 \lg n)$$

$$\sum_{i=n}^{\infty} 1/i^2 = \underline{\qquad} (1)$$

$$\binom{n}{10} = \underline{\qquad} (n^{10})$$

$$8^{\lg n} = \underline{\qquad} (n^3)$$

Problem 13. Find asymptotic bounds for the following recurrences. You may assume that $T(n) = \Theta(1)$ for $n \leq 1$ in each part.

(a) $T(n) = 2T(\lfloor n/2 \rfloor) + \Theta(n)$

(b) $T(n) = 4T(\lfloor n/2 + \sqrt{n} \rfloor) + \Theta(n^2)$

(c) $T(n) = T(\lceil n/3 \rceil) + 2T(\lfloor n/4 \rfloor) + \Theta(n)$

Problem 14. In the following problem there is no direct connection between the two parts.

(a) Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of one month and six new pairs of rabbits at the age of two months and every month afterwards. None of the rabbits ever die or leave the island. Find a recurrence relation for the number of *pairs* of rabbits on the island n months after one newborn pair is left on the island. Do not solve the recurrence relation.

(b) Solve the following inhomogeneous recurrence relation:

$$f(n) = 6f(n-1) - 9f(n-2) + 2^n, n \geq 2$$