Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Tom Leighton and Santosh Vempala

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Problem Set 6 Solutions

Problem 1. [10 points]

The running time of an algorithm A is described by the recurrence $T(n) = 7T(n/2) + n^2$. A competing algorithm A' has a running time of $T'(n) = aT'(n/4) + n^2$. For what value of a is A' asymptotically faster than A?

Using the Akra-Bazzi generalization of the master theorem, we have that for T, $k = 1, a_1 = 7, b_1 = 1/2, g(n) = n^2$ and for T', $k = 1, a_1 = a, b_1 = 1/4, g(n) = n^2$. In both cases, $g'(n) = 2n$ which is polynomially bounded. In the first case (for T), we have that $7 \cdot (1/2)^p = 1$ or that $p = \log_2 7$. In the second case (for T'), we have that $a \cdot (1/4)^p = 1 \Rightarrow a \cdot (1/2)^{2p} = 1 \Rightarrow p = \frac{1}{2}\log_2 a$. This gives

$$T(n) = \Theta(n^{\log_2 7}(1 + \int_1^n \frac{u^2}{u^{\log_2 7+1}} du)) = \Theta(n^{\log_2 7})$$

and

$$T'(n) = \Theta(n^{(\log_2 a)/2}(1 + \int_1^n \frac{u^2}{u^{(\log_2 a)/2+1}} du)) = \Theta(n^{(\log_2 a)/2}(1 + \int_1^n u^{1-(\log_2 a)/2} du)),$$

so that if $(\log_2 a)/2 > 2$, we have that $T'(n) = \Theta(n^{(\log_2 a)/2})$ and if $(\log_2 a)/2 = 2$, we have that $T'(n) = \Theta(n^{(\log_2 a)/2} \log n) = \Theta(n^2 \log n)$ and if $(\log_2 a)/2 < 2$, we have that $T'(n) = \Theta(n^2)$.

This means that for A' to be asymptotically faster than A, we need to have $\frac{1}{2}\log_2 a < \log_2 7$, or that $a < 7^2$. Hence, $a < 49$.

Problem 2. [30 points]

Use the formula given in class (Akra-Bazzi) to find tight asymptotic bounds (Θ) for the following divide-and-conquer recurrences. For each recurrence, $T(1) = 1$ and $T(n) = \Theta(1)$ for all constant n. State the value of p you get for each recurrence (which can be left in the form of logs). Also, state the values of the $a_i, b_i,$ and $h_i(n)$ for each recurrence.

1. $T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$
2. $T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n^2.$
3. $T(n) = 3T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n.$

4. $T(n) = T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$
5. $T(n) = T\left(\left\lceil \frac{n}{4} \right\rceil\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + n.$
6. $T(n) = 2T(\lfloor n/4 \rfloor) + \sqrt{n}.$
7. $T(n) = 2T(\lfloor n/4 \rfloor + 1) + \sqrt{n}.$
8. $T(n) = 2T(\lfloor n/4 + \sqrt{n} \rfloor) + 1.$
9. $T(n) = 3T(\left\lceil n^{1/3} \right\rceil) + \log_3 n.$
10. $T(n) = \sqrt{e}T(\left\lfloor n^{1/e} \right\rfloor) + \ln n.$

1. $a_1 = 3,\ b_1 = 1/3,\ h_1(n) = \lfloor n/3 \rfloor - n/3,\ g(n) = n,\ p = 1,$
$$T(n) = \Theta(n(1 + \int_1^n \frac{u}{u^2} du)) = \Theta(n \log n).$$
2. $a_1 = 4,\ b_1 = 1/3,\ h_1(n) = \lfloor n/3 \rfloor - n/3,\ g(n) = n^2,\ p = \log_3 4,$
$$T(n) = \Theta(n^{\log_3 4}(1 + \int_1^n \frac{u^2}{u^{\log_3 4 + 1}} du)) = \Theta(n^{\log_3 4}(1 + \int_1^n u^{1 - \log_3 4} du)) = \Theta(n^2).$$
3. $a_1 = 3,\ b_1 = 1/4,\ h_1(n) = \lfloor n/4 \rfloor - n/4,\ g(n) = n,\ p = \frac{1}{2}\log_2 3,$
$$T(n) = \Theta(n^{(\log_2 3)/2}(1 + \int_1^n \frac{u}{u^{(\log_2 3)/2 + 1}} du)) = \Theta(n^{(\log_2 3)/2}(1 + \int_1^n u^{-(\log_2 3)/2} du)) = \Theta(n).$$
4. $a_1 = 1, a_2 = 1,\ b_1 = 1/4,\ b_2 = 1/3,\ h_1(n) = \lfloor n/4 \rfloor - n/4,\ h_2(n) = \lfloor n/3 \rfloor - n/3,\ g(n) = n,$ $p = 0.5605$ (the exact value does not matter, it turns out that for this case, we only need to know that $p < 1$),
$$T(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) = \Theta(n^p(1 + \int_1^n u^{-p} du)) = \Theta(n).$$
5. $a_1 = 1,\ a_2 = 1,\ b_1 = 1/4,\ b_2 = 3/4,\ h_1(n) = \lceil n/4 \rceil - n/4,\ h_2(n) = \lfloor 3n/4 \rfloor - 3n/4,$ $g(n) = n,\ p = 1,$
$$T(n) = \Theta(n(1 + \int_1^n \frac{u}{u^2} du)) = \Theta(n \log n).$$
6. $a_1 = 2,\ b_1 = 1/4,\ h_1(n) = \lfloor n/4 \rfloor - n/4,\ g(n) = \sqrt{n},\ p = 1/2,$
$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{\sqrt{u}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$
7. $a_1 = 2,\ b_1 = 1/4,\ h_1(n) = \lfloor n/4 \rfloor - n/4 + 1,\ g(n) = \sqrt{n},\ p = 1/2,$
$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{\sqrt{u}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$

8. $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = \lfloor n/4 + \sqrt{n} \rfloor - n/4$, $g(n) = 1$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{1}{u^{3/2}} du)) = \Theta(n^{1/2}).$$

9. **NOTE: For this part the initial condition should have been $\mathbf{T(2) = 1}$**

We can turn this recurrence into a divide and conquer recurrence with a change of variables. First we show that $T(n)$ is non-decreasing.

Claim 0.1 *For all $n \geq 3$, $T(n) \geq T(n-1)$.*

Proof. By strong induction on n.
I.H.: $P(n) : T(n) \geq T(n-1)$.
Base: $n = 3$: $T(3) = 3T(\left\lceil 3^{1/3} \right\rceil) + 1 = 3T(2) + 1 = 4 > 1 = T(1)$.
Ind. Step: Assume $P(k)$ is true for all $3 \leq k \leq n$. We will prove that $P(n+1)$ is true.

$$\begin{aligned} T(n+1) &= 3T(\left\lceil (n+1)^{1/3} \right\rceil) + \log_3(n+1) \\ &\geq 3T(\lceil n \rceil) + \log n \\ &= T(n) \end{aligned}$$

So $T(n+1) \geq T(n)$. ■

We will now solve the recurrence assuming n is a power of 3. First, make the substitution $n = 3^m$. The recurrence becomes

$$T(3^m) = 3T(\left\lceil 3^{m/3} \right\rceil) + m$$

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Now define $S(m) = T(\lceil 3^m \rceil)$ for all integers $m \geq 0$. Note that since T is non-decreasing, for all $m > 1$, $S(m+1) \geq S(m)$, and hence S is also a non-decreasing function.
Therefore, (remember m is an integer),

$$\begin{aligned} S(m) &= T(\lceil 3^m \rceil) \\ &= T(3^m) \\ &= 3T(\left\lceil 3^{m/3} \right\rceil) + m \\ &\leq 3T(3^{\lceil m/3 \rceil}) + m \\ &\leq 3S(\lceil m/3 \rceil) + m \end{aligned}$$

That is $S(m) \leq 3S(\lceil m/3 \rceil) + m$

We can then view this as $S(m) \le 3S(m/3 + h(m)) + m$ where $h(m) = \lceil m/3 \rceil - m/3$. Clearly $h(m) < 1$ and hence it is certainly $h(m) = O(\frac{m}{(\log m)^2})$.
Now to solve this we can apply the Akra-Bazzi theorem to

$$S(m) = 3S(\frac{m}{3}) + m,$$

for which $a_1 = 3$, $b_1 = 1/3$, $h_1(m) = 0$, $g(m) = m$, $p = 1$,

$$S(m) = \Theta(m(1 + \int_1^m \frac{u}{u^2} du)) = \Theta(m \log m),$$

so that

$$T(n) \le T(3^{\lceil \log n \rceil}) = S(\lceil \log n \rceil) = O(\log n \log \log n).$$

Now we derive a lower bound in a similar fashion.

$$\begin{aligned} S(m) &= T(\lceil 3^m \rceil) \\ &= T(3^m) \\ &= 3T(\left\lceil 3^{m/3} \right\rceil) + m \\ &\ge 3T(3^{\lceil m/3 \rceil}) + m \\ &\ge 3S(\lceil m/3 \rceil) + m \end{aligned}$$

That is $S(m) \ge 3S(\lceil m/3 \rceil) + m$.
Define a new recurrence $S'(m) = 3S'(\lceil m/3 \rceil) + m$ with $S'(0) = 1$.

Claim 0.2 $S(m) \ge S'(m)$

Proof. By strong Induction on m.
I.H. : $P(m) : S(m) \ge S'(m)$.
Base : $m = 1$: $S(0) = S'(0) = 1$.
Ind. Step: Assume $P(k)$ is true for all $k \le m$. We will prove $P(m + 1)$.

$$\begin{aligned} S(m) &\ge 3S(\lceil m/3 \rceil) + m \\ &\ge 3S'(\lceil m/3 \rceil) + m \\ &\ge S'(m) \end{aligned}$$

∎

Now we can apply Akra-Bazzi to solve the recurrence for S' in a similar way as for the upper bound to get $S'(m) = \Theta(m \log m)$. From the claim above this implies that $S(m) = \Omega(m \log m)$.

Hence $T(n) = \Omega(\log n \log\log n)$. From the upper and lower bounds we conclude that $T(n) = \Theta(\log n \log\log n)$.

10. This problem is similar to that of part 9. We make the substitution $n = e^m$ and set $S(m) = T(e^m)$. Then the recurrence becomes

$$S(m) = \sqrt{e}S(\frac{m}{e}) + m,$$

for which $a_1 = \sqrt{e}$, $b_1 = 1/e$, $h_1(m) = 0$, $g(m) = m$, $p = 1/2$,
We follow the steps in part 9 to conclude that,

$$S(m) = \Theta(m^{1/2}(1 + \int_1^m \frac{u}{u^{3/2}}du)) = \Theta(m),$$

so that

$$T(n) = S(\log n) = \Theta(\log n).$$

Problem 3. [10 points]

Find an exact closed form solution for the recurrence

$$T(n) = \sum_{i=1}^{n-1} iT(i)$$

for $n > 1$ and $T(1) = 1$. (*Hint:* Find a simple formula for $T(n) - T(n-1)$.)

$$\begin{aligned} T(n) - (n-1)T(n-1) &= \sum_{i=1}^{n-2} iT(i) = T(n-1) \Rightarrow \\ T(n) &= (n-1)T(n-1) + T(n-1) \Rightarrow \\ T(n) &= nT(n-1) \end{aligned}$$

$T(1) = 1,\ T(2) = 1 \cdot T(1) = 1,$ and $T(3) = 2 \cdot T(2) + 1 \cdot T(1) = 3$. So we have:

$$T(n) = \begin{cases} 1 & \text{for } n = 1 \\ n!/2 & \text{for } n \geq 2 \end{cases}$$

Verify by induction:

Base case: $n = 2$

$T(2) = \sum_{i=1}^{1} iT(i) = 1 \cdot T(1) = 1 = 2!/2.$

Inductive step:

$$\begin{aligned} T(n+1) &= (n+1)T(n) \\ &= (n+1)\, n!/2 \qquad \text{by the induction hypothesis} \\ &= (n+1)!/2 \end{aligned}$$

Problem 4. [30 points]

Find an exact closed form solution for the following recurrences.

(a)

$$T(n) = 4T(n-1) - 4T(n-2) + n^2$$

for $n > 1$, and $T(0) = 0$, $T(1) = 1$.

First we find the general solution of the homogeneous recurrence:
$T(n) = 4T(n-1) - 4T(n-2)$.
Its characteristic equation is: $\alpha^2 - 4\alpha + 4 = 0 \Longrightarrow (\alpha - 2)^2 = 0$. So 2 is a root of multiplicity 2. The general solution has the form $A2^n + Bn2^n$ for $A, B \in \mathbb{R}$.
Now we find a particular solution to the non-homogeneous recurrence (ignoring initial conditions). For a polynomial "drive", we try a polynomial of the same degree (higher if 1 is a root of characteristic equation). Guess $T(n) = an^2+bn+c$, and solve for a, b, and c satisfying the recurrence:

$$\begin{aligned} an^2 + bn + c &= 4(a(n-1)^2 + b(n-1) + c) - 4(a(n-2)^2 + b(n-2) + c) + n^2 \\ &= 4((n-1)^2 - (n-2)^2) + 4b((n-1) - (n-2)) + 4c(1-1) + n^2 \\ &= 4a(2n-3) + 4b + n^2 \\ &= n^2 + 8an + 4b - 12c \end{aligned}$$

which implies that $a = 1,\ b = 8a = 8,\ c = 4b - 12a = 32 - 12 = 20$.
So a general solution to the inhomogeneous recurrence is $A2^n+Bn2^n+n^2+8n+20$ for $A, B \in \mathbb{R}$.
Finally we compute A, B using the initial conditions:
$0 = T(0) = A + 20 \Rightarrow A = -20$.
$1 = T(1) = 2A+2B+1+8+20 = 2B-40+29 \Rightarrow B = 1+40-29 = 12 \Rightarrow B = 6$.
Solution: $T(n) = (6n-20)2^n + n^2 + 8n + 20$.
Check answers for small values of n (to avoid arithmetic and algebraic errors):
$T(0) = -20 + 20 = 0$
$T(1) = -14 \cdot 2 + 1 + 8 + 20 = -28 + 29 = 1$
$T(2) = -8 \cdot 4 + 4 + 16 + 20 = -32 + 40 = 8 = 4 \cdot 1 - 4 \cdot 0 + 2^2$.

(b) Consider the recurrence
$T(n) = \sum_{i=1}^{d} a_i T(n/2^i) + G(n)$
Find a transformation $n \to m$ that will convert this problem in a difference equation of the form
$f(m) = \sum_{i=1}^{d} a_i f(m-i) + g(m)$

Let $n = 2^m$.
$T(2^m) = \sum_{i=1}^{d} a_i T(2^m/2^i) + G(2^m) = \sum_{i=1}^{d} a_i T(2^{m-i}) + G(2^m)$
Set $f(m) = T(2^m)$ and $g(m) = G(2^m)$.

(c) Apply your results from part (b), to find the solution of the recurrence relation
$T(n) = 3T(n/2) - 2T(n/4) + 3^{\log_2 n}$

Applying the transformation in (b), we obtain the recurrence
$f(m) = 3f(m-1) - 2f(m-2) + 3^m$.
The homogeneous part of this recurrence has characteristic equation
$x^2 - 3x + 2$
The roots are 2 and 1, so a solution has the form $c_1(1)^m + c_2 2^m = c_1 + c_2 2^m$.
To obtain a particular solution we try $k3^m$ to obtain:
$k(3^2) - 3k(3) + 2k = 3^2 \to k = 9/2$.
So the general solution to the linear recurrence is:
$f(m) = c_1 + c_2 2^m + (9/2)3^m$
Transforming back to the original form for $n = \log_2 m$ we obtain
$T(n) = c_1 + c_2 n + (9/2)3^{log_2 n}$

Problem 5. **[20 points]**

Bounty hunters in the Old West used the following empirical model: the number of outlaws caught (dead or alive) in a given year is the average of the number caught in the previous two years. Let's try out this model in the following hypothetical situation — in year 0, there were 100 outlaws caught, and in year 1, there were 300 outlaws caught.

(*note that this model allows fractional outlaws*)

(a) Let $f(n)$ denote the number of outlaws caught in year n. Write a recurrence relation for $f(n)$ and obtain a closed-form solution. Do you ever get a fractional outlaw?

$$f(n) = \frac{1}{2}(f(n-1) + f(n-2))$$

This is a homogeneous linear recurrence. We solve the characteristic equation, $2x^2 - x - 1 = 0$, to get the roots 1 and $-1/2$. Hence the general solution is of the form $f(n) = A + B(-1/2)^n$. Using the initial conditions $f(0) = 100$ and $f(1) = 300$ we find that the general solution is
$f(n) = 700/3 - (400/3)(-1/2)^n$.
The first fractional number of outlaws is obtained when $n = 3$.

(b) Now let's consider the following variation on this model: the number of outlaws caught in a given year is the average of the number caught in the previous *three* years. To start things off, suppose that 100 were caught in year 0, 300 in year 1, and 200 in year 2.

Let $g(n)$ denote the number of outlaws caught in year n under this model. Again, write a recurrence relation for $g(n)$ and obtain a closed-form solution.

$$g(n) = \frac{1}{3}(g(n-1) + g(n-2) + g(n-3))$$

This is a homogeneous linear recurrence. We solve the characteristic equation, $3x^3 - x^2 - x - 1 = 0 \Rightarrow (x-1)(3x^2 + 2x + 1) = 0$, to get the roots 1, $\frac{-1-i\sqrt{2}}{3}$ and $\frac{-1+i\sqrt{2}}{3}$ where $i = \sqrt{-1}$ denotes the imaginary unit. Hence the general solution is of the form
$g(n) = A + B(\frac{-1-i\sqrt{2}}{3})^n + C(\frac{-1+i\sqrt{2}}{3})^n$.
Using the initial conditions $g(0) = 100$, $g(1) = 300$ and $g(2) = 200$ we find that the general solution is

$$g(n) = \frac{650}{3} + \frac{-25}{3}(7 - i4\sqrt{2})\left(\frac{-1-i\sqrt{2}}{3}\right)^n + \frac{-25}{3}(7 + i4\sqrt{2})\left(\frac{-1+i\sqrt{2}}{3}\right)^n$$