Problem Set 7

Due: October 21.

Reading: Sections 1.6, 1.7 (pages 76-78), 4.2, of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 1. [10 points]

Determine whether each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, is an injection, surjection or a bijection. Also, give the inverse if it exists.

(*Note: The inverse of a function is defined in Rosen page 64.*)

- $f(x) = e^x$
- $f(x) = x^2 + 3$
- $f(x) = (x + 1)/(x^2 + 2)$
- $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$

Problem 2. [10 points]

Prove that the composition of two injective functions is injective.

(*Note: The composition of two functions is defined in Rosen, page 66.*)

Problem 3. [10 points]

Let A and B be finite sets.

(a) Prove that if $|A| \geq |B|$, then there exists a surjection from A to B.

(b) Prove that if $|A| \leq |B|$, then there exists an injection from A to B.

Problem 4. [20 points]

(a) **[10 points]** If $\mathcal{P}$ is any set of $n + 1$ integers selected from $1, 2, 3, \ldots, 2n$, prove that there are two relatively prime integers in $\mathcal{P}$.

(b) **[10 points]** Consider a set of k integers: $a_1, a_2, \ldots, a_k$. Show that there is some subset of these terms whose sum is divisible by k.

Problem 5. **[10 points]**

Suppose that you have a scale on which you can compare any two groups of balls and produce three outcomes (lighter, heavier, or equal). You are given five balls. All but one have the same weight. You are supposed to find which ball differs and how (lighter or heavier). Can you do this with two weighings?

Problem 6. **[10 points]** Let $\mathcal{S}_i, 1 \leq i \leq \infty$ be a family of countable sets. Prove that $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is countable.

Problem 7. **[10 points]**

Use a diagonalization argument to show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Problem 8. **[10 points]**

Let F be the set of all functions $f : \mathbb{R} \to \{0, 1\}$. Show that there is a bijection from $2^{\mathbb{R}}$ to F.

Problem 9. **[10 points]**

A real number is *algebraic* if it is the root of a polynomial with rational number coefficients. Algebraic numbers form an important part in the study of (what else) modern algebra. A number that is not algebraic is called *transcendental.*

(a) Prove that for any particular d, the set of degree d polynomials with rational coefficients is countable.

(b) Prove that the set of all roots of degree d polynomials with rational coefficients is countable (you may use the fact that a degree d polynomial has at most d roots).

(c) Prove that the set of algebraic numbers (which is just the union of the sets in the previous part) is countable (*Hint: use the result of problem 6*).

(d) Prove that there is a transcendental number.