Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Tom Leighton and Santosh Vempala

Handout 18
October 10, 1997

Problem Set 5 Solutions

Problem 1. **[10 points]**

A college loan officer offers you the following deal. He will give you \$22,000 at the start of each year for four years starting today to pay for your college tuition. In return, you must pay \$10,000 at the start of each year for twelve years starting in four years. The annual interest rate is 8%. Compute the present value of the loan and of the return. Is this a fair deal?

Present value of what the college loan officer offers

$$\begin{aligned}
&= 22000 + 22000/1.08 + 22000/(1.08)^2 + 22000/(1.08)^3 \\
&= 22000(1 - \frac{1}{(1.08)^4})/(1 - \frac{1}{1.08}) \\
&= 22000 * 3.577 \\
&= \$78,696.13
\end{aligned}$$

Present value of what you must pay in return

$$\begin{aligned}
&= 10000/(1.08)^4 + 10000/(1.08)^5 + \ldots + 10000/(1.08)^{15} \\
&= (10000/(1.08)^4)(1 - \frac{1}{(1.08)^{12}})/(1 - \frac{1}{1.08}) \\
&= \$59,823.82
\end{aligned}$$

This is a very fair deal for you!

Problem 2. **[10 points]**

Compute the following limits.

1. $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$.
2. $\lim_{n\to\infty} 1/n + 1/(n+1) + \ldots + 1/2n$

1. Using partial fractions, we find that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$.

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= 1 + \sum_{i=2}^{n} \frac{1}{i} - \sum_{i=2}^{n+1} \frac{1}{i} \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

   Taking the limit in the above expression we conclude that $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$.

2. $1/n + 1/(n+1) + \ldots + 1/2n) \leq \int_{n-1}^{2n} \frac{1}{n} dn = \ln n|_{n-1}^{2n} = \ln(\frac{2n}{n-1})$
   $1/n + 1/(n+1) + \ldots + 1/2n) \geq \int_{n}^{2n+1} \frac{1}{n} dn = \ln n|_{n}^{2n+1} = \ln(\frac{2n+1}{n})$
   $\lim_{n\to\infty} \ln(\frac{2n+1}{n}) = \ln 2$
   $\lim_{n\to\infty} \ln(\frac{2n}{n-1}) = \ln 2$
   The upper and lower bounds converge to $\ln 2$.

Problem 3. [10 points]

Rank the following functions by order of growth; that is find an arrangement $g_1 = \alpha(g_2) = \alpha(g_3) = \ldots = \alpha(g_6)$ and state whether α is o or Θ for each step.

- $n(\log_2 n)^5$
- n^3
- $n!$
- $(n!)^2$
- n^n
- $3 \cdot 8^{\log_2 n} + 4^{\log_2 n}$

$$n(\log_2 n)^5 = o(n^3) = \Theta(3 \cdot 8^{\log_2 n} + 4^{\log_2 n}) = o(n!) = o(n^n) = o((n!)^2).$$

To get the last two equalities we use Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n}, \text{ where } \frac{1}{12n+1} \leq a_n \leq \frac{1}{12n}$$

We have

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n}}{n^n} = \sqrt{2\pi n}\left(\frac{1}{e}\right)^n e^{a_n}$$

Taking the limit $n \to \infty$ in the above expression we have $e^{a_n} \to 1$ and $\sqrt{2\pi n}\left(\frac{1}{e}\right)^n \to 0$ because $n = o(e^n)$. Therefore the whole expression goes to 0. We conclude that $n! = o(n^n)$.

To get the last equality, we have:

$$\frac{n^n}{(n!)^2} = \frac{n^n}{(2\pi n)\left(\frac{n}{e}\right)^{2n} e^{2a_n}} = \frac{1}{(2\pi n)\left(\frac{n}{e^2}\right)^n e^{2a_n}}$$

The last expression goes to 0 as $n \to \infty$, so we conclude that $n^n = o((n!)^2)$.

Problem 4. [10 points]

Suppose that $f(x) = O(g(x))$.

(a) Does it follow that $2^{f(x)} = O(2^{g(x)})$?

(b) Does it follow that $f(x)^2 = O(g(x)^2)$?

Prove your answer to both parts.

1. This does not follow. Let $f(x) = 2x$ and $g(x) = x$. Then $f(x) = O(g(x))$. Now $2^{f(x)} = 2^{2x} = 4^x$, and $2^{g(x)} = 2^x$, and $4^x \neq O(2^x)$. Indeed, $\frac{4^x}{2^x} = 2^x$, so the ratio grows without bound as x grows– it is not bounded by a constant.
2. Yes. Since $f(x) = O(g(x))$, $\exists x_0, c$ such that $\forall x \geq x_0, |f(x)| \leq cg(x)$. So, $\forall x \geq x_0, f(x)^2 \leq c^2 g(x)^2$. Therefore, there exist $x_0' = x_0$ and $c' = c^2$ such that $\forall x \geq x_0', f(x)^2 \leq c' g(x)^2$. So $f(x)^2 = O(g(x)^2)$

Problem 5. [10 points]

Give asymptotically tights bounds for the following summations.

(a) $\sum_{k=1}^{n} k^5$

(b) $\sum_{k=1}^{n} \ln^2 k$

(c) $\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$

1.

$$\begin{aligned}\sum_{k=1}^{n} k^5 &\leq \int_1^{n+1} x^5 dx \\ &= \frac{(n+1)^6 - 1}{6} \\ &\sim n^6/6\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} k^5 &\geq \int_0^{n} x^5 dx \\ &= \frac{n^6}{6} \\ &\sim n^6/6\end{aligned}$$

It follows that $\sum_{k=1}^{n} k^5 = \Theta(n^6/6)$.

2.

$$\begin{aligned}\sum_{k=1}^{n} \ln^2 k &\leq \int_1^{n+1} \ln^2 x dx \\ &= \int_0^{ln(n+1)} u^2 e^u du \\ &= e^{\ln(n+1)} \ln^2(n+1) + \text{lower order stuff} \\ &= O(n \ln^2 n)\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} \ln^2 k &\geq \int_1^{n} \ln^2 x dx \\ &= \int_0^{ln(n)} u^2 e^u du \\ &= e^{\ln(n)} \ln^2(n) + \text{lower order stuff} \\ &= \Omega(n \ln^2 n)\end{aligned}$$

It follows that $\sum_{k=1}^{n} \ln^2 k = \Theta(n \ln^2 n)$.

3.

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &\leq \int_0^{n} \frac{1}{\sqrt{x}} dx \\ &= 2\sqrt{n} \\ &\sim 2\sqrt{n}\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} &\geq \int_{1}^{n+1}\frac{1}{\sqrt{x}}dx\\ &= 2(\sqrt{n+1}-1)\\ &\sim 2\sqrt{n}\end{aligned}$$

It follows that $\sum_{k=1}^{n}\frac{1}{\sqrt{k}} = \Theta(2\sqrt{n})$.

Problem 6. [10 points]

Given that $x < 1$ and $y < 1$, express the following in closed form

$$\prod_{p=1}^{\infty}\prod_{q=p}^{\infty} e^{-\frac{pq}{2}x^{p/2}y^{q}}$$

$$\prod_{p=1}^{\infty}\prod_{q=p}^{\infty} e^{-\frac{pq}{2}x^{p/2}y^{q}} =$$
$$e^{-\sum_{p=1}^{\infty}\sum_{q=p}^{\infty}\frac{pq}{2}x^{p/2}y^{q}}$$

Now we compute $\sum_{p=1}^{\infty}\sum_{q=p}^{\infty}\frac{pq}{2}x^{p/2}y^{q}$.

$$\begin{aligned}\sum_{q=p}^{\infty}\frac{pq}{2}x^{p/2}y^{q} &= \frac{p}{2}x^{p/2}\sum_{q=p}^{\infty}qy^{q}\\ &= \frac{p}{2}x^{p/2}\left(\sum_{q=1}^{\infty}qy^{q}-\sum_{q=1}^{p-1}qy^{q}\right)\\ &= \frac{p}{2}x^{p/2}y\left(\frac{py^{p-1}}{1-y}-\frac{y^{p}}{(1-y)^{2}}\right)\end{aligned}$$

We know from tutorial that

$$\sum_{k=1}^{\infty}k^{2}a^{k} = \frac{2a^{2}}{(1-a)^{3}}-\frac{a}{(1-a)^{2}}$$

We will use this to compute the sum over p.

$$\begin{aligned}
\sum_{p=1}^{\infty}\sum_{q=p}^{\infty}\frac{pq}{2}x^{p/2}y^{q} &= \sum_{p=1}^{\infty}\frac{p}{2}x^{p/2}y\left(\frac{py^{p-1}}{1-y}-\frac{y^{p}}{(1-y)^2}\right)\\
&= \frac{1}{2(1-y)}\sum_{p=1}^{\infty}p^2(yx^{1/2})^p-\frac{y}{2(1-y)^2}\sum_{p=1}^{\infty}p(yx^{1/2})^p\\
&= \frac{1}{2(1-y)}\left(\frac{2xy^2}{(1-yx^{1/2})^2}-\frac{yx^{1/2}}{(1-yx^{1/2})^2}\right)-\frac{y}{2(1-y)^2}\frac{yx^{1/2}}{(1-yx^{1/2})^2}
\end{aligned}$$

Therefore the final result is

$$\prod_{p=1}^{\infty}\prod_{q=p}^{\infty}e^{-\frac{pq}{2}x^{p/2}y^{q}} = exp\left(\frac{1}{2(1-y)}\left(\frac{2xy^2}{(1-yx^{1/2})^2}-\frac{yx^{1/2}}{(1-yx^{1/2})^2}\right)-\frac{y}{2(1-y)^2}\frac{yx^{1/2}}{(1-yx^{1/2})^2}\right)$$

Problem 7. [10 points]

Show that

(a) $\lfloor x+1/2\rfloor = \Theta(x)$.

(b) $\log(x^2+1) = \Theta(\log_2 x)$.

(c) $\log_{10} x = \Theta(\log_2 x)$.

(d) $\log(n!) = \Theta(n\log n)$.

1. For $x > 2$ we certainly have $\lfloor x+1/2\rfloor \le 2x$, and also $x \le 2\lfloor x+1/2\rfloor$.
2. For $x > 2$, $\log(x^2+1) \le \log(2x^2) = 1+2\log x \le 3\log x$ (recall that log means $\log_2$). On the other hand, since $x < x^2+1$ for all positive x, we have $\log x \le \log(x^2+1)$.
3. This follows from the fact that $\log_{10} x = C(\log_2 x)$, where $C = 1/\log_2 10$.
4. Clearly $n! \le n^n$, $\forall\ n \ge 1$. Therefore, $\log(n!) \le \log(n^n) = n\log n$. Thus, $\log(n!) = O(n\log n)$.

   Approximate $\log(n!)$ with *Stirling's formula* for $n!$ to get:

$$\begin{aligned}
\log(n!) &= \log(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n}) \text{ where } \tfrac{1}{12n+1} \le a_n \le \tfrac{1}{12n}\\
&= \log\sqrt{2\pi n}+\log n^n-\log e^n+\log e^{a_n}\\
&= \log\sqrt{2\pi n}+n\log n-n\log e+a_n\log e\\
&\ge n\log n-n\log e\\
&\ge \frac{1}{2}n\log n \text{ since } n\log e \le \tfrac{n}{2}\log n \text{ for } n \ge e^2
\end{aligned}$$

   Hence, $\log(n!) = \Omega(n\log n)$, which together with part (a) implies that $\log(n!) = \Theta(n\log n)$.

Problem 8. [10 points]

Recall that the harmonic numbers are given by $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Prove that

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

We could use induction, or the following.

$$\begin{aligned}
\sum_{k=1}^{n} H_k &= \sum_{k=1}^{n}\sum_{j=1}^{k} \frac{1}{j} \\
&= \sum_{1 \le j \le k \le n} \frac{1}{j} \\
&= \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{1}{j} \\
&= \sum_{j=1}^{n} \frac{n-j+1}{j} \\
&= \sum_{j=1}^{n} \left(\frac{n+1}{j} - 1 \right) \\
&= (n+1)\sum_{j=1}^{n} \frac{1}{j} - n \\
&= (n+1)H_n - n
\end{aligned}$$

Problem 9. [10 points]

Find a closed form expression for $\sum_{i=1}^{n} i^3$. Prove it is correct.

By bounding the above summation with integrals we can conclude that $\sum_{i=1}^{n} i^3 \sim n^4/4$.

We try an expression of the form $n^4/4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0$.

By setting $n = 0$, we conclude that $a_0 = 0$. By computing the values of the summation for $n = 1, 2, 3$ we obtain

$$\begin{array}{lr}
n = 1 & 1 = 1/4 + a_3 + a_2 + a_1 \\
n = 2 & 9 = 1/4 + 8a_3 + 4a_2 + 2a_1 \\
n = 3 & 36 = 1/4 + 27a_3 + 9a_2 + 3a_1
\end{array}$$

Solving this system of linear equations we obtain $a_1 = 0$, $a_2 = 1/4$, and $a_3 = 1/2$.

so our guess is $\sum_{i=1}^{n} i^3 = n^4/4 + n^3/2 + n^2/4 = (n(n+1)/2)^2$. We prove it by induction.

Base case: $n = 1$

$1 = ((1 \cdot 2)/2)^2$

Inductive step:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= (n(n+1)/2)^2 + (n+1)^3 \\
&= \frac{(n+1)^2}{4}(n^2 + 4(n+1)) \\
&= \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}$$

The induction is complete.

Problem 10. [10 points]

Suppose that $f(n) = \Theta(n^{1.5})$. Find a tight theta bound for $\sum_{i=1}^{n} f(i)$. Prove that your answer is correct.

For some constants c_1, c_2 and $n_0 \in \mathbb{N}$, we have that $\forall n \geq n_0, c_1 n^{1.5} < f(n) < c_2 n^{1.5}$.

Let L denote the constant $L = f(1) + \ldots + f(n_0)$.

$$\begin{aligned}
\sum_{i=1}^{n} f(i) &\leq c_2 \sum_{i=1}^{n} n^{1.5} + f(1) + \ldots + f(n_0) \\
&\leq c_2 \int_1^{n+1} x^{1.5} dx + L \\
&= c_2 \frac{(n+1)^{2.5} - 1}{2.5} + L \\
&\leq c_2 2^{2.5} n^{2.5} + L \\
&\leq (c_2 2^{2.5} + L) n^{2.5}
\end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{n} f(i) &\geq c_1 \sum_{i=1}^{n} n^{1.5} + f(1) + \ldots + f(n_0) \\
&\geq c_1 \int_0^n x^{1.5} dx + L \\
&= c_1 \frac{n^{2.5}}{2.5} + L \\
&\geq \frac{c_1}{2.5} n^{2.5}
\end{aligned}
$$

We have found constants $c_1' = \frac{c_1}{2.5}$ and $c_2' = c_2 2^{2.5} + L$ such that $c_1' n^{2.5} \leq \sum_{i=1}^{n} f(i) \leq c_2' n^{2.5}$. We conclude that $\sum_{i=1}^{n} f(i) = \Theta(n^{2.5})$.