Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 17
October 7, 1997

Problem Set 6

Due: October 14, 1997.

Reading assignment: 5.1-5.3 in Rosen

Problem 1. **[10 points]**

The running time of an algorithm A is described by the recurrence $T(n) = 7T(n/2) + n^2$. A competing algorithm A' has a running time of $T'(n) = aT'(n/4) + n^2$. For what value of a is A' asymptotically faster than A?

Problem 2. **[30 points]**

Use the formula given in class (Akra-Bazzi) to find tight asymptotic bounds (Θ) for the following divide-and-conquer recurrences. For each recurrence, $T(1) = 1$ and $T(n) = \Theta(1)$ for all constant n. State the value of p you get for each recurrence (which can be left in the form of logs). Also, state the values of the $a_i, b_i,$ and $h_i(n)$ for each recurrence.

1. $T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$
2. $T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n^2.$
3. $T(n) = 3T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n.$
4. $T(n) = T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$
5. $T(n) = T\left(\left\lceil \frac{n}{4} \right\rceil\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + n.$
6. $T(n) = 2T(\lfloor n/4 \rfloor) + \sqrt{n}.$
7. $T(n) = 2T(\lfloor n/4 \rfloor + 1) + \sqrt{n}.$
8. $T(n) = 2T(\lfloor n/4 + \sqrt{n} \rfloor) + 1.$
9. $T(n) = 3T(\left\lceil n^{1/3} \right\rceil) + \log_3 n.$
10. $T(n) = \sqrt{e}T(\left\lfloor n^{1/e} \right\rfloor) + \ln n.$

Problem 3. **[10 points]**

Find an exact closed form solution for the recurrence

$$T(n) = \sum_{i=1}^{n-1} iT(i)$$

for $n > 1$ and $T(1) = 1$. (*Hint:* Find a simple formula for $T(n) - T(n-1)$.)

Problem 4. **[30 points]**

Find an exact closed form solution for the following recurrences.

(a)
$$T(n) = 4T(n-1) - 4T(n-2) + n^2$$
for $n > 1$, and $T(0) = 0$, $T(1) = 1$.

(b) Consider the recurrence
$T(n) = \sum_{i=1}^{d} a_i T(n/2^i) + G(n)$
Find a transformation $n \to m$ that will convert this problem in a linear recurrence of the form
$f(m) = \sum_{i=1}^{d} a_i f(m-i) + g(m)$

(c) Apply your results from part (b), to find the exact solution of the recurrence relation
$T(n) = 3T(n/2) - 2T(n/4) + 3^{\log_2 n}$

Problem 5. **[20 points]**

Bounty hunters in the Old West used the following empirical model: the number of outlaws caught (dead or alive) in a given year is the average of the number caught in the previous two years. Let's try out this model in the following hypothetical situation — in year 0, there were 100 outlaws caught, and in year 1, there were 300 outlaws caught.

(*note that this model allows fractional outlaws*)

(a) Let $f(i)$ denote the number of outlaws caught in year i. Write a recurrence relation for $f(i)$ and obtain a closed-form solution. Do you ever get a fractional outlaw?

(b) Now let's consider the following variation on this model: the number of outlaws caught in a given year is the average of the number caught in the previous *three* years. To start things off, suppose that 100 were caught in year 0, 300 in year 1, and 200 in year 2.