Problem Set 5

Due: Tuesday, October 7.

Reading assignment: Rosen 1.7-1.8

Problem 1. [10 points]

A college loan officer offers you the following deal. He will give you \$22,000 at the start of each year for four years starting today to pay for your college tuition. In return, you must pay \$10,000 at the start of each year for twelve years starting in four years. The annual interest rate is 8%. Compute the present value of the loan and of the return. Is this a fair deal?

Problem 2. [10 points]

Compute the following limits.

1. $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$.
2. $\lim_{n\to\infty} 1/n + 1/(n+1) + \ldots + 1/2n$

Problem 3. [10 points]

Rank the following functions by order of growth; that is find an arrangement $g_1 = \alpha(g_2) = \alpha(g_3) = \ldots = \alpha(g_6)$ and state whether α is o or Θ for each step.

- $n(\log_2 n)^5$
- n^3
- $n!$
- $(n!)^2$
- n^n
- $3 \cdot 8^{\log_2 n} + 4^{\log_2 n}$

Problem 4. [10 points]

Suppose that $f(x) = O(g(x))$.

(a) Does it follow that $2^{f(x)} = O(2^{g(x)})$?

(b) Does it follow that $f(x)^2 = O(g(x)^2)$?

Prove your answer to both parts.

Problem 5. [10 points]

Give asymptotically tights bounds for the following summations.

(a) $\sum_{k=1}^{n} k^5$

(b) $\sum_{k=1}^{n} \ln^2 k$

(c) $\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$

Problem 6. [10 points]

Given that $x < 1$ and $y < 1$, express the following in closed form

$$\prod_{p=1}^{\infty} \prod_{q=p}^{\infty} e^{-\frac{pq}{2} x^{p/2} y^q}$$

Problem 7. [10 points]

Show that

(a) $\lfloor x + 1/2 \rfloor = \Theta(x)$.

(b) $\log(x^2 + 1) = \Theta(\log_2 x)$.

(c) $\log_{10} x = \Theta(\log_2 x)$.

(d) $\log(n!) = \Theta(n \log n)$.

Problem 8. [10 points]

Recall that the harmonic numbers are given by $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Prove that

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

Problem 9. [10 points]

Find a closed form expression for $\sum_{i=1}^{n} i^3$. Prove it is correct.

Problem 10. **[10 points]**

Suppose that $f(n) = \Theta(n^{1.5})$. Find a tight theta bound for $\sum_{i=1}^{n} f(i)$. Prove that your answer is correct.