Massachusetts Institute of Technology | Handout 13
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Problem Set 3 Solutions

Problem 1. [7 points]

Let G be a bipartite graph with n vertices and m edges. Show that $m \leq n^2/4$

Suppose the parts are of sizes k and $n-k$. Then the maximum number of edges the graph may have is $k(n-k)$. The maximum of the function $f(k) = k(n-k)$ is achieved at $k = n/2$, and has the value $f(n/2) = n^2/4$. Therefore, $m \leq n^2/4$.

Problem 2. [7 points]

Find the adjacency matrix for the graphs K_n, C_n and W_n defined in 7.3 of Rosen.

In K_n every vertex is connected to every other vertex, so the adjacency matrix has zeros in the diagonal and 1's everywhere else.

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

For C_n, we label the vertices so that the cycle goes $v_1, v_2, \dots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's everywhere else.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

The adjacency matrix of W_n is similar to the adjacency matrix for C_n, except that we add one more row and column for the vertex in the middle of the wheel (which is connected to everything but itself).

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{pmatrix}$$

Problem 3. [15 points]

Let $G = (V, E)$ be an undirected graph, and let there be a weight $r(v)$ associated with each vertex v. Suppose r satisfies the property that for each $v \in V$, the value $r(v)$ is the average of the values of v's neighbors. Prove that if the graph is connected, then $r(u) = r(v)$ for all $u, v \in V$.

Proof.

The proof is by contradiction.

Suppose that not all the vertices in the graph have the same value. Let $s = \min_{u \in V}(u)$. Note that this value may not be unique in the graph. Choose a vertex v such that $r(v) = s$ and v has a neighbor u with $r(u) > s$. We will prove that such a vertex exists.

Pick a vertex x_0 of value s and a vertex w with $r(w) > s$ (such a vertex exists because of the assumption that r is not constant). Since G is connected, there exists a path $(x_0, x_1, x_2, \dots, x_{n-1}, x_n)$ with $x_n = w$. Let i be the first number from 0 to $n-1$ such that $r(x_i) < r(x_{i+1})$. Then the vertex x_i is a vertex with value s that has a neighbor of greater value. Set $v = x_i$ and $u = x_{i+1}$.

Now, $r(u) > r(v)$ implies that the average of the values of the neighbors of v must be greater than s. Since $r(v) = s$ is supposed to be equal to this average we obtain a contradiction. We conclude that r has to be constant.

■

Problem 4. [7 points]

Prove that in any undirected graph, the sum of the squares of the vertex degrees is even, or provide a counterexample.

The assertion is true.

Proof 1

Let V_e be the set of even degree vertices and V_o the set of odd degree vertices of the graph. From class we know that there is an even number of odd degree vertices. So can write

$$\sum_{v \in V} deg(v)^2 = \sum_{v \in V_e} deg(v)^2 + \sum_{v \in V_o} deg(v)^2 \tag{1}$$

If $v \in V_e$ then $deg(v)$ is even, hence $deg(v)^2$ is also even. Therefore, the first term in the right hand side of (1) is even. Similarly if $v \in V_o$, then $deg(v)^2$ is odd. The second term in the right hand side is a sum of odd numbers with an even number of terms. Therefore it is also even. We conclude that $\sum_{v \in V} deg(v)^2$ is even

Proof 2

Let E be the set of edges of the graph and $V = \{v_1, \ldots, v_n\}$ be the set of vertices of the graph.

$$(\sum_{i=1}^{n} deg(v_i))^2 = \sum_{i=1}^{n} deg(v_i)^2 + 2\sum_{i \neq j} deg(v_i)deg(v_j) \tag{2}$$

The left hand side of (2) is even because $\sum_{i=1}^{n} deg(v_i) = 2|E|$ and the second part of the right hand side is obviously even. Therefore $\sum_{i=1}^{n} deg(v_i)^2$ is even.

Problem 5. [7 points]

Prove that any tree with a degree k vertex has at least k leaves.

First we prove the following lemma

Lemma: A connected subgraph of a tree is itself a tree.

Proof. A tree T is a connected graph with unique paths between any two vertices. Let H be a connected subgraph of T. Since H is connected there is at least one path between any two vertices in H. If there are more than one paths connecting two vertices of H, then there are more than one paths connecting two vertices of G which is a contradiction. ■

Now we prove the statement $P(k) =$ "A tree with a degree k vertex has at least k leaves" by strong induction on k.

Base case: $P(1)$ is true because a degree one vertex is by definition a leaf.

Inductive step: Consider a tree with a degree $k+1$ vertex v. If v is connected to a leaf, remove that leaf. The resulting tree has a degree k vertex therefore it has at least k leaves. Re-attach the leaf. The initial tree has $k+1$ leaves.

If v is not connected to a leaf, then v is connected to $k+1$ vertices $v_1, \ldots, v_{k+1}$ each being adjacent to at least one edge other than the one connecting them with v. Let $H_1, \ldots, H_{k+1}$ be the subgraphs obtained by detaching v_i from v. Each of the H_i is connected, because there is a path from any vertex of H_i to v_i, therefore, by the lemma, they are subtrees. Furthermore, they are disjoint. For if there is a path between v_i and v_j, then there are two paths connecting v_i and v_j, since v_i and v_j are also connected through v. This contradicts the tree definition.

Now, since none of the $v_1, \ldots, v_{k+1}$ is a leaf, each of the trees $H_1, \ldots, H_{k+1}$ has a vertex of degree at least 1, so by the inductive hypothesis, each has at least one leaf. Since the $H_1, \ldots, H_{k+1}$ are disjoint, the leaves are distinct. Therefore the original tree has at least $k+1$ leaves.

Problem 6. [7 points]

Let G be a graph with n vertices and m edges. Let $dmax$ be the maximum degree of the vertices of G, and let $dmin$ be the minimum degree of the vertices of G. Show that

(a) $2m/n \geq dmin$.

First observe that $2m$ is the sum of the degrees of the vertices (each edge contributes a degree of 2 since it is incident to two vertices). But the total degree cannot be less than the minimum degree multiplied by the number of vertices. This means that $2m \geq n \cdot dmin$ or that $2m/n \geq dmin$.

(b) $2m/n \leq dmax$.

As before, observe that $2m$ is the sum of the degrees of the vertices. Since the total degree cannot be greater than the maximum degree multiplied by the number of vertices. This means that $2m \leq n \cdot dmax$ or that $2m/n \leq dmax$.

Problem 7. [10 point]

You are given the following two facts.

F1) Any subgraph of a planar graph is planar.

F2) Any planar graph with n vertices and m edges satisfies $m \leq 3n - 6$.

(a) Use (F2) to show

(F3) Any planar graph has a vertex of degree at most 5.

Suppose that every vertex has degree at least 6. Then $2m = \sum_{v \in V} deg(v) \geq 6n \rightarrow m \geq 3n$ which contradicts (F2).

(b) Use F1 and F3 to show that any planar graph can be colored with six colors.

The proof is by induction on the number of vertices n.

$P(n) =$ "A planar graph of n vertices can be colored with at most 6 colors".

Base case: $P(1)$ is true because a single vertex can be colored with 1 color.

Inductive step: Assume $P(n)$ is true in order to show that $P(n+1)$ is true.

Let G be a planar graph with $n+1$ vertices and remove a vertex v of degree 5 (or less). The remaining n-vertex graph can be colored with 6 colors by the inductive hypothesis. Re-attach v. v is adjacent to at most 5 vertices, occupying at most 5 out of the six colors. We can use the remaining color to color v.

We have shown that $P(n) \rightarrow P(n+1)$, so the proof is complete.

Problem 8. [15 points]

Show that a directed multigraph has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

First we will show that if the multigraph has an Euler circuit, then the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

Suppose that the directed multigraph has an Euler circuit. Since the circuit provides a path from any vertex to any other vertex, the graph is strongly connected (and therefore weakly connected). Now, by the existence of the Euler circuit, every vertex is entered as many times as it is exited. Since every time a vertex is entered 1 is added to the in-degree count and every time it is exited 1 is added to the out-degree count, the degrees must have the same value.

Conversely, suppose that we have a directed multigraph G which is weakly connected and the in-degree and out-degree of each vertex are equal. We argue by contradiction. Suppose that it doesn't have an Euler circuit and let C be a maximum-length cycle (allowing repeated vertices but not repeated edges) in the graph. If C contains all the edges in G we are done. If not, there is an edge e that is not in C. Since G is connected, there is a path from e that eventually reaches a vertex in C. This means that there is an edge e' that is not in C and is incident to a vertex v in C. Now consider the subgraph $G - C$. Note that v is a vertex of $G - C$. Also note that for any vertex in C, the in-degree has been decreased by 1 every time it was entered and the out-degree decreased by 1 any time it was exited. Therefore the in and out degrees of the vertices in $G - C$ are still equal.

Start with vertex v and construct a path in $G-C$ as follows. We know that v is adjacent to some edge e', and since the in and out degrees of v are equal, there is at least one out-edge connecting v to some vertex v_1. We let (v, v_1) be the first vertex of the path and we proceed similarly from v_1 to construct a path $C' = \{v, v_1, \ldots, v_k\}$ as long it is possible without repeating any edges.

Note that v_k can only be v. Otherwise, due to the fact that the in-degree and out-degree are equal, there should be an edge leading out of v_k. We conclude that C' is a cycle.

Now consider the cycle in the original graph G that starts at v goes around C and then goes around C'. This cycle contains at least one more edge than C (namely e') so the maximality of C is contradicted.

The proof is complete.

Problem 9. [10 points]

We provide an alternative definition for a tree.

Definition: A *tree* is a connected graph such that $\forall u, v \in V$, there is a unique path connecting u to v.

Prove that this definition is equivalent to the one that defines a tree to be an acyclic graph of n vertices that has $n-1$ edges.

Let G be a graph of n vertices such that $\forall u, v \in V$, there is a unique path connecting u to v.

First we show that G is acyclic. Suppose, for the purposes of contradiction, that G has a cycle. Pick two vertices u and v on the cycle. Then there are at least two paths connecting v and u, contradicting the uniqueness assumption. So G is acyclic.

Also we note that the existence of paths between every two vertices implies that G is connected. So G is a connected acyclic graph, and from class we know that it has $n-1$ edges.

Conversely, let G be an acyclic graph with n vertices and $n-1$ edges. We know from class that such a graph is connected so there exists at least one path between any two vertices.

Suppose, for the purposes of contradiction, that there are two distinct paths C and C' connecting the vertices u and v. Starting at u consider the first vertex u' where C differs from C' (such a vertex exists because the graphs are distinct). Also consider the first vertex v' where C and C' meet again (such a vertex exists because both paths end at v). Then there is a simple cycle that starts at u' follows C until it reaches v' and returns to u' following C'. This contradicts the acyclicity of the graph.

Problem 10. [15 points]

Definition: Let S be a set. Let R be a subset of $S \times S$. R is an *equivalence relation* if it satisfies the following properties

1. $\forall a \in S, (a, a) \in R$ (reflexivity).
2. If $(a, b) \in R$, then $(b, a) \in R$ (symmetry).
3. If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ (transitivity).

Definition: Let R be an equivalence relation on S. For $a \in S$, we define the *equivalence class* of a to be the set $[a]_R = \{b \in S | (a, b) \in R\}$.

For example, the relation R defined by $(a, b) \in R$ if $a \equiv b \bmod 4$ is an equivalence relation on $\mathbb{Z}$. There are four equivalence classes for this relation, namely:

- $[0]_R = \{z \in \mathbb{Z} | z \equiv 0 \bmod 4\}$
- $[1]_R = \{z \in \mathbb{Z} | z \equiv 1 \bmod 4\}$
- $[2]_R = \{z \in \mathbb{Z} | z \equiv 2 \bmod 4\}$
- $[3]_R = \{z \in \mathbb{Z} | z \equiv 3 \bmod 4\}$

(a) Let G be a simple graph. Define a relation R on the set of vertices V of G by $(v, u) \in R$, if there is a path connecting u to v.

Verify that R is an equivalence relation.

- Reflexivity:
  R is reflexive because trivially every vertex is connected to itself.
- Symmetry:
  Suppose there is a path $(u, x_1), (x_1, x_2), \ldots, (x_n, v)$ connecting u to v. Then, the path $(v, x_n), \ldots, (x_1, u)$ connects v to u. Therefore, R is symmetric.
- Transitivity:
  Let the path $(u, x_1), (x_1, x_2), \ldots, (x_n, v)$ connect u to v, and the path $(v, y_1), (y_1, y_2), \ldots, (y_m, w)$ connect v to w.
  Then the path $(u, x_1), \ldots, (x_n, v), (v, y_1), \ldots, (y_m, w)$ connects u to v.
  Therefore R is transitive.

(b) What are the equivalence classes of R?

The equivalence classes of R are the connected components of G.