Problem Set 4

Due: Tuesday, September 30.

Reading assignment: Chapter 6 of Rosen, 2.1-2.4 and 3.3-3.5. Section 2.5 is interesting but optional

Problem 1. [**40 points**]

In this problem you are asked to design a protocol for college admissions. The protocol must have the following properties:

There are N students $S_1, ..., S_N$ and M universities $U_1, ..., U_M$. Each student provides a preference ranking of all M universities and each university provides a preference ranking of all N students. For each i university U_i has n_i slots to fill and sum of all slots is equal to the number of students. That is, $\sum_{i=1}^{M} n_i = N$. The protocol should assign every student to a university so that

1. For each i University U_i gets n_i students,
2. There does not exist students S_i, S_j and universities U_k, U_l where S_i is assigned to U_k and S_j is assigned to U_l, but S_j prefers U_k to U_l, and U_k prefers S_j to S_i.
3. The assignment algorithm should be student optimal in the sense that every student gets their top choice among all assignments that satisfy conditions 1 and 2 above.

(a) [**8 points**] Model the problem as a state machine. State clearly what the states and state transitions are. What is the start state?

(b) [**8 points**] Use derived variables to prove termination. How long does the protocol take to terminate?

(c) [**8 points**] Prove that every student is assigned and every university gets the right number of students.

(d) [**8 points**] Prove that there are no assignments violating condition 2.

(e) [**8 points**] Prove that condition 3 holds.

Problem 2. [**20 points**]

Consider the following multiplication algorithm ,known as the "Russian peasant's multiplication"

x, y, are arbitrary positive integers whose values are to be multiplied. a, is the accumulator for the answer.

r, s, are temporary variables.

Initially, $r = x$, $s = y$, $a = 0$. We denote the next value of r by r', etc...

Do until $s = 0$

If s is even and $s > 0$,
$r' := 2r$
$s' := \frac{s}{2}$

If s is odd,
$a' := a + r$
$r' := 2r$
$s' := \frac{s-1}{2}$

The answer is the value in the accumulator a when $s = 0$.

Here is an example of the execution of the algorithm.

\$x\$	\$y\$	\$r\$	\$s\$	\$a\$
5	9	5	9	0
		10	4	5
		20	2	5
		40	1	5
		80	0	45

Model the computation as a state machine. Prove it terminates using a derived variable. Prove that it is correct using an invariant.

Problem 3. [10 points]

Evaluate $\gcd(F_n, F_{n-1})$ for any n, where F_n is the nth Fibonacci number.

Problem 4. [10 points]

Derive an example of preferences for the unisex version of the mating problem (all boys or all girls) which show that it is not possible to always pair all the players so that the marriages satisfy the stability criterion.

(Hint: 4 people is enough)

Problem 5. [20 points]

Luke Skywalker is using a training device (like the one in Starwars I) to develop his light-saber fencing skills. The training device is a contraption that in every time step makes a move in a three dimensional grid and emits a non-lethal laser beam. This particular training device can only move as follows: At every step it can move 1 unit north or south AND 1 unit east or west AND 1 unit up or down. Luke is using the Force to guide his feelings but also common sense to guard against possible attack points.

If the device starts at position $(0, 0, 0)$, should Luke ever guard against an attack coming from position $(1, 1, 2)$? Prove your answer using state machines and invariants.

(Note: There are many invariants that will work for this problem.)