Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 10
September 16, 1997

Problem Set 3

Due: Tuesday, September 23.

Reading assignment: 7.1-7.5,7.7 of Rosen, 7.8 optional

Problem 1. [**7 points**]

Let G be a bipartite graph with n vertices and m edges. Show that $m \leq n^2/4$

Problem 2. [**7 points**]

Find the adjacency matrix for the graphs K_n, C_n and W_n defined in 7.3 of Rosen.

Problem 3. [**15 points**]

Let $G = (V, E)$ be an undirected graph, and let there be a weight $r(v)$ associated with each vertex v. Suppose r satisfies the property that for each $v \in V$, the value $r(v)$ is the average of the values of v's neighbors. Prove that if the graph is connected, then $r(u) = r(v)$ for all $u, v \in V$.

Problem 4. [**7 points**]

Prove that in any undirected graph, the sum of the squares of the vertex degrees is even, or provide a counterexample.

Problem 5. [**7 points**]

Prove that any tree with a degree k vertex has at least k leaves.

Problem 6. [**7 points**]

Let G be a graph with n vertices and m edges. Let $dmax$ be the maximum degree of the vertices of G, and let $dmin$ be the minimum degree of the vertices of G. Show that

(a) $2m/n \geq dmin$.

(b) $2m/n \leq dmax$.

Problem 7. **[10 point]**

You are given the following two facts.

F1) Any subgraph of a planar graph is planar.

F2) Any planar graph with n vertices and m edges satisfies $m \leq 3n - 6$.

(a) Use (F2) to show

(F3) Any planar graph has a node of degree at most 5.

(b) Use F1 and F3 to show that any planar graph can be colored with six colors.

Problem 8. **[15 points]**

Show that a directed multigraph has an Euler circuit if and only if the graph is weakly connected (see Rosen p.465 definition 5), and the in-degree and out-degree of each vertex are equal.

Problem 9. **[10 points]**

We provide an alternative definition for a tree.

Definition: A *tree* is a connected graph such that $\forall u, v \in V$, there is a unique path connecting u to v.

Prove that this definition is equivalent to the one that defines a tree to be an acyclic graph of n vertices that has $n - 1$ edges.

Problem 10. **[15 points]**

Definition: Let S be a set. Let R be a subset of $S \times S$. R is an *equivalence relation* if it satisfies the following properties

1. $\forall a \in S, (a, a) \in R$ (reflexivity).
2. If $(a, b) \in R$, then $(b, a) \in R$ (symmetry).
3. If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ (transitivity).

Definition: Let R be an equivalence relation on S. For $a \in S$, we define the *equivalence class* of a to be the set $[a]_R = \{b \in S | (a, b) \in R\}$.

For example, the relation R defined by $(a, b) \in R$ if $a \equiv b \bmod 4$ is an equivalence relation on $\mathbb{Z}$. There are four equivalence classes for this relation, namely:

- $[0]_R = \{z \in \mathbb{Z} | z \equiv 0 \text{ mod } 4\}$
- $[1]_R = \{z \in \mathbb{Z} | z \equiv 1 \text{ mod } 4\}$
- $[2]_R = \{z \in \mathbb{Z} | z \equiv 2 \text{ mod } 4\}$
- $[3]_R = \{z \in \mathbb{Z} | z \equiv 3 \text{ mod } 4\}$

(a) Let G be simple a graph. Define a relation R on the set of vertices V of G by $(v, u) \in R$, if there is a path connecting u to v.

Verify that R is an equivalence relation.

(b) What are the equivalence classes of R?