# Problem Set 2

**Due: Tuesday, September 16.**

**Reading assignment:** Finish reading the *Nuts and Bolts of Proofs*. Read 3.2 of Rosen.

**Problem 1.** Prove by induction that for every $n \in \mathbb{N}$, either $4|n^2$ or $4|(n^2 - 1)$.

**Problem 2.** Use mathematical induction to prove that $\forall n \geq 4$, $n^2 \leq 2^n$ (*Note: If you use a fact such as $n^2 \geq 2n + 1$ for $n \geq 4$, don't forget to prove it*)

**Problem 3.** What are the fallacies in the following arguments? Don't just explain why the claim is wrong; rather, explain how the argument violates the notion of a correct proof.

**(a)** Claim: Every natural number is a prime or a perfect square.

Proof: as the inductive hypothesis, use $P(n) =$"every number less than or equal to $n$ is a prime or a perfect square."

$P(1)$ is certainly true. Now consider $n$. If $n$ is prime we are done. Otherwise, $n$ can be factored as $n = rs$ with $r$ and $s$ less than or equal to $n - 1$. By the inductive hypothesis, $r$ and $s$ are perfect squares, $r = u^2$ and $v = s^2$. Then $n = rs = u^2v^2 = (uv)^2$.

**(b)** *Claim*: $\log_3 n = \log_2 n$ for all natural numbers $n$.

Proof by strong induction:

The inductive hypothesis is $P(n) =$ "$\log_3 n = \log_2 n$".

*base case*: $\log_3 1 = 0 = \log_2 1$ is true.

*inductive step*: Assume that $\log_3 k = \log_2 k$ for all natural numbers $k \leq n$, and show it is true for $n+1$. Write $n+1$ as a product of two natural numbers $p$ and $q$ so that we have:

$$\log_3(n+1) = \log_3(pq) = \log_3 p + \log_3 q = \log_2 p + \log_2 q = \log_2(pq) = log_2(n+1)$$

which is true by inductive hypothesis.

**Problem 4.** Use mathematical induction to prove that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4.$$

**Problem 5.** Define the *Nifobacci numbers* $N_i$, by

$N_0 = 0$

$N_1 = 1$

$N_{n+1} = {N_n}^2 + {N^2}_{n-1}$.

Prove that $N_n$ is even if and only if $3|n$.

**Problem 6.** The National Bank of Aldouria produces coins with denominations of either 4 or 7 Aldourian pounds. A sum of money is called *round* if it can be made up with Aldourian coins.

**(a)** Characterize the Aldourian round sums.

**(b)** Prove that any amount of money exceeding 17 is round using induction.

**Problem 7.** For what values of $m$ and $n$ can you tile an $n \times m$ courtyard if you are given tiles that are $3 \times 1$ and $2 \times 2$? Prove that your answer is correct.

(*Note:* $3 \times 1$ *tiles can be rotated to form* $1 \times 3$ *tiles*).

**Problem 8.** Use induction to prove that $\forall n \in \mathbb{N}$ and any $n$ sets $A_1, A_2, \ldots A_n$,

$$(A_1 \cup A_2 \cup \ldots A_n)^c = A_1^c \cap A_2^c \cap \ldots A_n^c$$

**Problem 9.** Prove that $k!|n(n+1)(n+2)\cdots(n+k-1)$ for all $k, n \in \mathbb{N}$.

**Problem 10.** An *n-player tournament* is a collection of two-player games such that each of the $n$ players competes with each of the other players exactly once. (Thus there are $n(n-1)/2$ games). A player can either win or lose in a single game (there are no ties).

Prove that in any $n$-player tournament, if there is no player that wins all his/her matches, then there is a collection of players $p_1, p_2, \ldots, p_k$ (for some $k \leq n$), such that player $p_i$ beats player $p_{i+1}$ for $1 \leq i < k$, and player $p_k$ beats player $p_1$.