Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 54
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Problem Set 11

Due: December 04, 1995.

Problem 11-1. Let S be a sample space. Say random variable $X : S \to \mathbb{N}$ has distribution

$$X = \begin{cases} 2 & \text{with probability } 1/4, \\ 3 & \text{with probability } 1/4, \\ 5 & \text{with probability } 1/2; \end{cases}$$

and random variable $Y : S \to \mathbb{N}$ has distribution

$$Y = \begin{cases} 1 & \text{with probability } 2/3, \\ 4 & \text{with probability } 1/3; \end{cases}$$

where X and Y are independent.

(a) What is the expected value of $X + Y$?

(b) What is the expected value of $\max(X, Y)$?

(c) What is the expected value of $\min(X, Y)$ (there's a quick answer from the other two parts)?

Problem 11-2. A couple keeps having babies until they get a child of each gender, whereupon they stop. Each child is equally likely to be a boy or a girl. What is the expected number of children that they will have?

Problem 11-3. You are given a wheel (like a roulette wheel) which contains the numbers from 1 to $2n$ arranged in some way. The wheel has a spinner and on each spin you get the two numbers at the opposite ends of the spinner. How would you arrange the numbers on the wheel to maximize the expected value of:

(a) The sum of the numbers chosen?

(b) The product of the numbers chosen?

Problem 11-4. We are given a random vector of n distinct numbers. We then determine the maximum of these numbers by the following procedure:

Pick the first number. Call it the "current maximum". Go through the rest of the vector (in order) and each time we find a number x that exceeds our "current maximum", we update the "current maximum" with x.

What is the expected number of times we update the "current maximum"?

Problem 11-5. Deviations from the mean:

(a) Prove that if X is a random variable with mean m, then $X - m$ is a random variable with mean 0

(b) Suppose I flip n unbiased independent coins. We know that the mean number of heads is $n/2$. What is the *expected deviation from the mean*? That is, if the number of heads is a random variable X, what is

$$E[|X - n/2|]?$$

Problem 11-6. (A true story from world war two). Suppose the army needs to test its n soldiers for a disease that occurs independently at random with probability p for each soldier. The test is by means of a blood test. One approach (i) is to test each soldier individually: this requires n tests. An alternative approach (ii) is the following. Group the soldiers into groups of k. Blend the blood samples of each group and apply the test once to each blended sample. If the group-blend is free of the disease, we are done after one test. If the group-blend fails the test, then someone has the disease and we have to test all k people for a total of $k + 1$ tests.

(a) What is the expected number of tests in method (ii) as a function of the number of soldiers n, the disease probability p, and the group size k?

(b) How should k be chosen to minimize the expected number of test performed, and what is the resulting expectation?

(c) What fraction of the work does method (ii) expect to save over method (i) in a million-strong army with disease incidence 1%?

(d) Can you come up with a better scheme by using multiple levels of grouping (ie, groups of groups)?

Problem 11-7. Consider a tournament among 10 teams, each playing the other. Recall that a *chain* is a ranking of the teams such that each team beats the one just behind it in the ranking. Prove that there is a tournament outcome such that there are over 3000 distinct chains. *Hint:* If each game outcome is random (50-50), what is the expected number of chains? Can all tournaments end up with fewer chains than the expected number?

This problem is an instance of the *probabilistic method.* It uses probability to prove the existence of an object without constructing it.