Problem Set 9

Due: November 14, 1995.

Problem 9-1. A function is said to be *non-decreasing* if $x \le y \Rightarrow f(x) \le f(y)$. How many non-decreasing functions are there from $\{1 \ldots n\}$ to $\{1 \ldots n\}$? (Hint: Think $*$'s and $|$'s)

Problem 9-2. How many n-bit sequences have more 1's than 0's?

Problem 9-3. Let $f(m, n)$ denote the number of different shortest paths from $(0, 0)$ to (m, n).

(a) Find a closed form for $f(m, n)$.

(b) Prove that $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m,n)x^m y^n = \frac{1}{1-x-y}$.

Problem 9-4. Prove that

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{9i^2-3i-2} = \frac{1}{3}$

(b) $\displaystyle\sum_{i=1}^{n} i \log i = \Theta(n^2 \log n)$

Problem 9-5. Prove by induction on n that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$. (Hint: use Pascal's identity)

Problem 9-6. Prove that

(a) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.

(b) $f(n) + o(f(n)) = \Theta(f(n))$.

Problem 9-7. Two numbers are said to be *coprime* if their greatest common divisor is 1. Find the number of numbers between 1 and 1728 coprime to 1728.

Problem 9-8. Give tight (Θ) bounds for the following recurrences:

(a) $T(n) = T(n/4) + T(4n/9) + 5n; T(n) = O(1), n \leq 10.$

(b) $T(n) = 2T(n-1) + 3T(n-2); \quad T(0) = T(1) = 1.$

Problem 9-9. Rank the following functions by order of growth; that is, find an arrangement $g_1, g_2, \ldots, g_5$ of the functions satisfying $g_i = O(g_{i+1})$. If $g_i = o(g_{i+1})$ or if $g_i = \Theta(g_{i+1})$ then mention that also.

(a) $\log n!$

(b) $n^{\log n}$

(c) $2^{\log n}$

(d) $2^{\sqrt{\log n}}$

(e) n^3.

Problem 9-10. A balloon-shop sells party balloons in different colours, called imaginatively $C_1, C_2, \ldots$. Balloons of colour C_i are sold only in packets of size 2^i. What's more, the store you go to is down to its last packet of each colour i.e. if you want colour C_4 you can buy 0 or 16 balloons.

(a) Prove that the generating function for the number of ways to have an assortment of n coloured balloons is $(1+x)(1+x^2)\ldots(1+x^{2^i})\ldots$.

(b) Show that $f(n)$ - the coefficient of x^n in the generating function of part a) - is 1 for all n.

(c) Conclude that $\frac{1}{1-x} = (1+x)(1+x^2)\ldots(1+x^{2^i})\ldots$.