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Notes 16

1 Generating Functions

Definition 1.1 *Let* $< g_0, g_1, g_2, \ldots >$ *be an infinite sequence. The (ordinary)* generating function *of the sequence is the formal power series*

$$G(x) = \sum_{n=0}^{\infty} g_n x^n \left(= \sum_n g_n x^n \textit{ simplified notation} \right).$$

Note that x is formal, i.e. symbolic. Do not actually evaluate $G(x)$, just use its structure.

Examples:

$$G(x) = \frac{1}{1-x} = \sum_n 1 \cdot x^n \longleftrightarrow < 1, 1, 1, \ldots >$$

$$G(x) = 1 + x \longleftrightarrow < 1, 1, 0, 0, 0, \ldots >$$

$$G(x) = (1+x)^r = \sum_n \binom{r}{n} x^n \longleftrightarrow < \binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \ldots, \binom{r}{r}, 0, 0, 0, \ldots >$$

$$< 1, a, a^2, a^3, \ldots > \longleftrightarrow \sum_n a^n x^n = \frac{1}{1-ax}$$

$$< 1, -1, 1, -1, 1, -1, \ldots > \longleftrightarrow \frac{1}{1+x} \text{ } (a = -1 \text{ above})$$

$$< 1, 0, 1, 0, 1, 0, \ldots > \longleftrightarrow \sum_n a^n x^{2n} = \sum_n a^n (x^2)^n = \frac{1}{1-x^2}$$

Or,

$$< 1, 0, 1, 0, 1, 0, \ldots > \longleftrightarrow \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) = \frac{1}{1-x^2}$$

$$< \underbrace{0, 0, 0, \ldots, 0}_{k}, 1, 1, 1, \ldots > \longleftrightarrow x^k \frac{1}{1-x}$$

$$< 1, 2, 3, 4, \ldots > \longleftrightarrow \sum_n (n+1)x^n = \frac{d}{dx}\sum_n x^n = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

Ordinary Generating Functions (OGF) represent all elements of a sequence as a single formula.

Two useful rules are:

OGF's are can be combined in linear fashion.

$$\begin{aligned} & < f_0, f_1, f_2, \ldots > \longleftrightarrow F(x) \text{ and } < g_0, g_1, g_2, \ldots > \longleftrightarrow G(x) \\ \Longrightarrow \quad & < \alpha f_0 + \beta g_0, \alpha f_1 + \beta g_2, \alpha f_2 + \beta g_2, \ldots > \longleftrightarrow \alpha F(x) + \beta G(x) \end{aligned}$$

Derivatives of OGF are also OGF.

$$< f_0, f_1, f_2, \ldots > \longleftrightarrow F(x) \Longrightarrow < f_1, 2f_2, 3f_3, 4f_4, \ldots > \longleftrightarrow F'(x)$$

Solving recurrenences using OGF's.

We can use OGF to solve many recurrences. For example, the Fibanocci recurrence is

$$< 0, 1, 1, 2, 3, 5, 8, 11, \ldots > \longleftrightarrow \frac{x}{1-x-x^2}$$

Recall, the Fibanocci recurrence is defined as

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} & \text{if } n \geq 2 \end{cases}$$

1. Define sequence and OGF of sequence.

$< a_0, a_1, a_2, \ldots >$

$$A(x) = \sum_n a_n x^n$$

2. Multiply both sides of recurrence by x^n and sum over valid indices.

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n$$

3. Express in terms of OGF

$$LHS = A(x) - a_0 - a_1 x = A(x) - x, \text{ since } a_0 = 0 \text{ and } a_1 = 1$$

$$\begin{aligned} RHS &= \sum_{n=0}^{\infty} (a_n + a_{n+1}) x^{n+2} \\ &= x^2 \sum_n a_n x^n + x \sum_n a_{n+1} x^{n+1} \\ &= x^2 A(x) + x(\sum_n a_n x^n - a_0) \\ &= x^2 A(x) + xA(x) \end{aligned}$$

Therefore,

$$A(x) - x = x^2 A(x) + xA(x)$$

4. Solve for $A(x)$

$$A(x)(1 - x - x^2) = x \Longrightarrow A(x) = \frac{x}{(1 - x - x^2)}$$

5. Solve for a_n = coefficient of x^n in $A(x)$. (Often use partial fractions if rational functions)

Using partial fraction decompositon, we see

$$A(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right)$$

where $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.

Coeff of x^n in $\frac{1}{1-\alpha_1 x}$ is α_1^n.

Coeff of x^n in $\frac{1}{1-\alpha_2 x}$ is α_2^n.

Therefore,

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

2 OGF's for Combinations

Recall,

$$< \binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \ldots, \binom{r}{r}, 0, 0, 0, \ldots > \longleftrightarrow (1+x)^r$$

Suppose we have r distinguishable objects $\{a_1, \ldots, a_r\}$.

What is the OGF of the number of ways to choose objects from $\{a_1\}$?

$g_0 = 1$

$g_1 = 1$

$g_2 = 0$

$g_3 = 0$

$g_4 = 0$

etc.

$G(x) = 1 + x$

What is the OGF of the number of ways to choose objects from $\{a_2\}$? Also, $1 + x$.

What is the OGF of the number of ways to choose objects from $\{a_1, a_2\}$?

$(1+x)(1+x) = (1+x)^2 = 1 + 2x + x^2$

In $(1 + x)$, the 1 represents not choosing a_i, the x represents choosing a_i.

Thus, we have 1 way to choose 0 elements, 2 ways to choose 1 element, and 1 way to choose two elements. The coefficient of x^n is the number of ways to choose n elements.

In general, $(1 + x)^r$ is the OGF for the number of ways to choose from $\{a_1, \ldots, a_r\}$.

Theorem 2.1 *Let $A(x)$ and $B(x)$ be OGF's for the number of ways to choose objects from sets $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then, the OGF for the number of ways of choosing objects from $\mathcal{A} \cup \mathcal{B}$ is $A(x) \cdot B(x)$.*

Proof. Let $A(x) = \sum_n a_n x^n$, $B(x) = \sum_n b_n x^n$.

Let $C(x) = A(x)B(x) = \sum_n c_n x^n$.

What is c_n?

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$< c_0, c_1, \ldots >$ is the *convolution sequence* of $< a_0, a_1, \ldots >$ and $< b_0, b_1, \ldots >$.

Now, how many ways is there to choose n objects from $\mathcal{A} \cup \mathcal{B}$? For $k = 0, 1, 2, \ldots, n$, pick k objects from $\mathcal{A}$ and $n-k$ objects from $\mathcal{B}$. Rule of product $\Longrightarrow a_k b_{n-k}$ ways for a given k.

Total $= \sum_{k=0}^{n} a_k b_{n-k} = c_n$. ∎

Fruit example; Consider choosing a bag of n fruits such that

- No. of apples is even
- No. of bananas is a multiple of 5
- No. of oranges is less than or equal to 4
- No. of pears is either 0 or 1.

How many different bags of fruit are there?

For $n = 6$, we can list the ways.

A	6	4	4	2	2	0	0
B	0	0	0	0	0	5	5
O	0	2	1	4	3	1	0
P	0	0	1	0	1	0	1

For arbitrary n, we can compute the generating function for each fruit and then multiply.

Apples: $A(x) = 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$.

Bananas: $B(x) = 1 + x^5 + x^{10} + \ldots = \frac{1}{1-x^5}$.

Oranges: $O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$.

Pears: $P(x) = 1 + x = \frac{1-x^2}{1-x}$.

Total OGF:

$$F(x) = A(x)B(x)O(x)P(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot \frac{1-x^2}{1-x} = \frac{1}{(1-x)^2} = \sum_n (n+1)x^n$$

The answer is $(n+1)$ ways for a bag of n fruits.

Another example: $\mathcal{A}$ has ∞ red objects. $\mathcal{B}$ has ∞ blue objects. Count the number of ways of picking n objects from $\mathcal{A}$ and $\mathcal{B}$.

OGF for the number of ways of picking red (blue) objects is $1 + x + x^2 + \ldots = \frac{1}{1-x}$.

OGF for the number of ways of picking red or blue objects is

$$\frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{1}{1-x^2} = \sum_n (n+1)x^n$$

Example: What is the OGF of the number of ways to choose objects from k types, where repeated types are allowed?

The OGF is $(1 + x + x^2 + \ldots)^k = (1 - x)^{-k}$

We can evaulate the coefficients two ways. One way is to use the generalized binomial theorem.

$$\begin{aligned}
(1-x)^{-k} &= \sum_n \binom{-k}{n} (-1)^n x^n \\
&= \sum_n \frac{(-k)(-k-1)\ldots(-k-n+1)}{n!} (-1)^n x^n \\
&= \sum_n \frac{(k)(k+1)\ldots(k+n-1)}{n!} x^n \\
&= \sum_n \binom{k+n-1}{n} x^n
\end{aligned}$$

Another way is to evaulate the terms in the Taylor series using derivatives. Recall, from Calculus, that the nth term of Taylor series, f_n, of the function $F(x)$ around 0 is $F^{(n)}(0)/n!$.

In our case,

$$F(x) = (1 - x)^{-k}$$

Thus,

$f_0 = F(0)/0! = 1$

$F'(x) = k(1 - x)^{-(k+1)} \Longrightarrow f_1 = F'(0)/1! = k$

$F''(x) = k(k + 1)(1 - x)^{-(k+2)} \Longrightarrow f_2 = F'(0)/2! = k(k + 1)/2$

In general,

$$\begin{aligned}
& F^{(n)}(x) = k(k+1)\ldots(k+n-1)(1-x)^{-(k+n)} \\
\Longrightarrow\quad & f_n = F'(0)/n! = k(k+1)\ldots(k+n-1)/n! = \tbinom{k+n-1}{n}.
\end{aligned}$$