Problem Set 8

Due: November 07, 1995.

Reading: Sections 4.6, 4.7, 5.4 and 5.5 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 8-1. **Multinomial Theorem.** Prove that for any variables x_i and positive integer n

$$\left(\sum_{i=1}^{k} x_i\right)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

(a) algebraically and

(b) combinatorially.

Problem 8-2. **Submultisets.**

(a) How many distinguishable submultisets of the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ are there?

(b) How many different divisors of 75600 are there?

Problem 8-3. **(a)** Consider the Towers of Hanoi game with n disks and 3 distinct poles. An arrangement of the disks on the poles is said to be legal if no disk rests on a smaller disk. How many different legal arrangements of the n disks on the 3 poles are there?

(b) How many distinguishable ways are there to put n books on r shelves (order on the shelf matters)?

Problem 8-4. 11 scientists are working on a secret project. They want to lock the documents in a cabinet so that the cabinet cannot be opened by any subset of five scientists. But every subset of six scientists should be able to open it.

(a) Prove that a minimum of $\binom{11}{5}$ different locks is needed on the cabinet.

(b) Prove that each scientist must carry at least $\binom{10}{5}$ keys.

Problem 8-5. Compute the (ordinary) generating function for each of the following sequences.

- **(a)** $\langle 0,0,0,1,1,2,1,1,1,1,1\ldots\rangle$
- **(b)** $\langle 0,0,0,1,2,3,4,5,6,7,8,9\ldots\rangle$
- **(c)** $\langle 1,0,1,0,1,0,1,\ldots\rangle$
- **(d)** $\langle 1,4,9,16,25,\ldots\rangle$
- **(e)** $\langle 2,6,12,20,30,42,56,\ldots\rangle$
- **(f)** $\langle a_k\rangle = \langle k(k+2)\rangle$
- **(g)** $\langle a_k\rangle = \left\langle 2^k + 3^k\right\rangle$
- **(h)** $\langle a_k\rangle = \langle 1/k!\rangle$
- **(i)** $\langle a_k\rangle = \left\langle \sum_{i=0}^{k} \frac{1}{i!} \right\rangle$
- **(j)** $\langle 1,1/2,1/3,1/4,\ldots\rangle$

Problem 8-6. Find partial fraction decompositions for the following functions.

- **(a)** $\dfrac{x^4+x^3+3x^2+6x+7}{x^2+3x+2}$
- **(b)** $\dfrac{x-6}{x^3+x^2-x-1}$

Problem 8-7. Use generating functions to solve the recurrence

$$a_n = \begin{cases} n & \text{if } 0 \le n < 3 \\ 2a_{n-2} + a_{n-3} & \text{if } n \ge 3 \end{cases}$$

Use induction to check your answer.