Massachusetts Institute of Technology | Handout 27
6.042J/18.062J: Mathematics for Computer Science | October 17, 1995
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Problem Set 6

Due: October 24, 1995.

Related Handouts: [MR] Chapter 5, [CLR] Chapters 2,3,4 .

Problem 6-1. A college loan officer offers you the following deal. He will give you \$25,000 at the start of each year for four years starting today to pay for your college tuition. In return, you must pay \$15,000 at the start of each year for fifteen years starting in four years. Suppose the annual interest rate is 5%. Is this a fair deal?

Problem 6-2. Find a closed form expression $f(n)$ for $\displaystyle\sum_{i=1}^{n} i^2 x^i$.

Problem 6-3. Let $S_n = \sum_{i=1}^{n} \sqrt{i}$. What is

$$\lim_{n\to\infty} \frac{S_n}{n^{1.5}}?$$

Problem 6-4. Use iteration to find a closed form expression for the recurrence

$$T(n) = aT(n/2) + bn$$

for $n > 1$ and $T(1) = b$, where a and b are constants. You may assume n is a power of 2. Consider separately the cases $a < 2$, $a = 2$, and $a > 2$.

Problem 6-5. Give tight asymptotic bounds (Θ) for the following recurrences. For each, $T(1) = 1$.

(a) $T(n) = 3T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n^2$.

(b) $T(n) = 2T\left(\left\lfloor \frac{n}{5} \right\rfloor\right) + n$.

(c) $T(n) = 4T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n$.

(d) $T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$.

Problem 6-6. Let $0 < x < 1$ and $0 < y < 1$. Evaluate the following infinite sums.

(a)

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^i y^{i+2j}.$$

(b)

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij x^i y^j.$$

Problem 6-7. Apparently, a standard model used by lobster fishermen is the following: the number of lobsters caught in a given year is the average of the number caught in the previous two years. Let's try out this model in the following hypothetical fishing community — in year 0, there were 100 lobsters caught, and in year 1, there were 300 lobsters caught.

(a) Let $f(i)$ denote the number of lobsters caught in year i. Write a recurrence relation for $f(i)$ and obtain a closed-form solution.

(b) Now let's consider the following variation on this model: the number of lobsters caught in a given year is the average of the number caught in the previous *three* years. To start things off, suppose that 100 were caught in year 0, 300 in year 1, and 200 in year 2.

Let $g(i)$ denote the number of lobsters caught in year i under this model. Again, write a recurrence relation for $g(i)$ and obtain a closed-form solution.

(c) Finally – let's go back to the model from the first part of this question (just averaging the previous two years, with initial conditions 100 lobsters in year 0 and 300 in year 1). But now there's a new complication in our imaginary fishing village — in year $i \geq 0$, i lobsters migrate into the fishing grounds from far-off waters, and all of these new lobsters get caught as well.

Let $h(i)$ denote the number of lobsters caught in year i under this model. Write a recurrence relation for $h(i)$ and obtain a closed-form solution. (*Hint:* For the inhomogeneous part of your solution, guess a function of the form $Ai^2 + Bi$.)