# Problem Set 5

**Due: Start of Class, October 17.**

**Reading:** Sections 6.6, 7.1–7.5, and 7.8 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

**Problem 5-1.** What does the Hasse diagram look like for a totally ordered set on $n$ elements. Prove your answer. (Hint: use induction and the lemma about the existence of a minimal element in any poset, that was proved in class.)

**Problem 5-2.** Consider a poset of $N$ tasks such that the length of the longest chain is $D$. You have $P$ processors on which to execute these tasks. Each task takes one unit of time to execute on a processor.

**(a)** Show that $D$ is a lower bound on the number of time units needed to execute all the tasks.

**(b)** Show that $N/P$ is also a lower bound on the number of time units needed to execute all the tasks.

**(c)** Show that all the tasks can be executed in time $D + N/P$. (Hint: consider a partition of the poset into $D$ antichains.)

**Problem 5-3.** **(a)** A set of circles on the plane is said to be *good* if for any pair of circles one is contained within the other. A set of circles on the plane is said to be *bad* if for any pair of circles neither one is contained within the other. Show that in any collection of $N$ circles there is a subset of $\sqrt{N}$ circles which is good or bad.

**(b)** A set of natural numbers is said to be *good* if for any pair of numbers one divides the other. A set of natural numbers is said to be *bad* if for any pair of numbers neither one divides the other. Show that in any collection of $N$ numbers there is a subset of $\sqrt{N}$ numbers which is good or bad.

(Note: if you plan to use a theorem about posets make sure to prove that the structure in question is in fact a poset.)

**Problem 5-4.** Consider the graph above.

**(a)** How many connected components does it have?

**(b)** List all the nodes in the graph along with their associated degrees.

**(c)** For each pair of vertices consider the number of paths of length 3 between them and represent this data in the form of a matrix. What relationship does this matrix bear to the adjacency matrix of the graph?

**Problem 5-5.** A graph is said to be bipartite if it is 2-colorable. Show that every tree is bipartite. (Hint: first show that every tree has a leaf.)

**Problem 5-6.** What is wrong with the following "theorem" and "proof?"
*False theorem:* If every vertex in a graph has degree at least one then the graph is connected.
*False proof:* We prove the result by induction on the number of vertices. Let $P(n)$ be "if every vertex in a graph with $n$ vertices has degree at least one then the graph is connected."
Base case. $P(2)$ is true since the only graph satisfying the assumption is that with two vertices and one edge between them which is easily seen to be connected.
Inductive step. Assume $P(n)$ is true. Now, take any graph on $n$ vertices with minimum degree at least one and add a new vertex to it. By inductive hypothesis the graph on $n$ vertices is connected. Since the new vertex satisfies the assumption it must have degree at least one, so it must have an edge connecting it to one of the remaining $n$ vertices. Thus the whole graph is connected. Hence $P(n+1)$ is true. Thus, $P(n)$ is true for all $n$.