Massachusetts Institute of Technology | Handout 18
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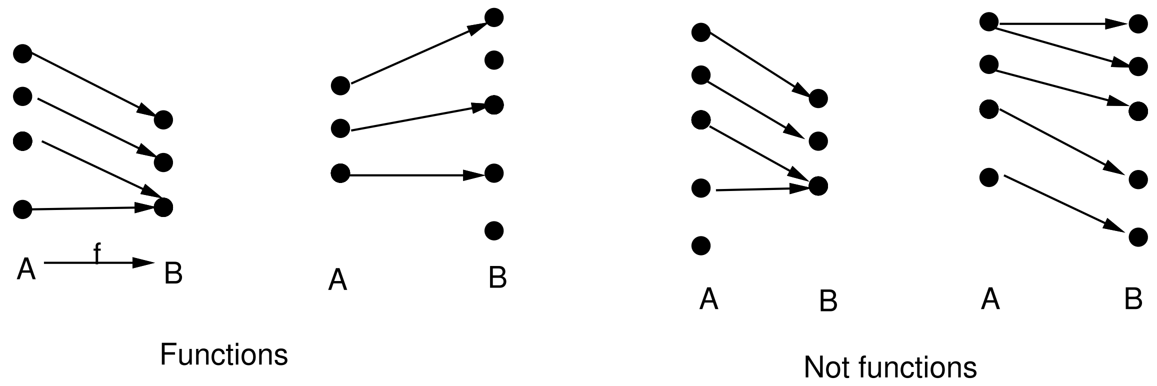
Lecture Notes 6
Functions, Pigeonhole Principle.

1 Functions

Definition 1.1 *A* function $f : A \to B$ *is a way of relating* every *item in a set, A, with* precisely one *item in a set, B.*

The set A is called the *domain* of f, and the set B is called the *range* of f.



To be a function, every item in A must be related to just one item in B. Note that every item in B need not be hit. The image of f is the set of points that are hit. Formally, the image of f, denoted by $f(A)$, is $\{b \in B | \exists a \in A \text{ s.t. } f(a) = b\}$.

You should be familiar with many functions such as $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = |x|$ and $g : \mathbb{R} \to \mathbb{Z}$ where $g(x) = \lfloor x \rfloor$.

An example of a non-function is $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \sqrt{x}$. This is not a function because $\sqrt{-1}$ is not a real. Even if we extend the range to the complex numbers, we still have the problem that $\sqrt{1} = \pm 1$. If we only take the positive root, then square root is no longer distributive over negative numbers.

Functions can also be defined recursively. The Fibonacci function, $f : \mathbb{N} \to \mathbb{N}$ can be defined as

$$f(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f(n-1) + f(n-2) & \text{if } n > 1. \end{cases}$$

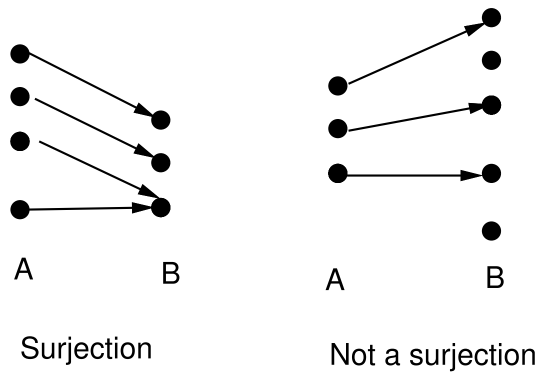
The factorial function can be similarly defined as

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ nf(n-1) & \text{if } n > 0. \end{cases}$$

Three important classifications of functions are surjections, injections, and bijections.

Definition 1.2 *A function $f : A \to B$ is a* surjection *(aka as onto) if $\forall b \in B, \exists a \in A$ such that $f(a) = b$.*

In a surjection, every element in B is hit by at least one element.



Definition 1.3 *A function $f : A \to B$ is an* injection *(aka as 1-1) if $f(a_1) = f(a_2)$, then $a_1 = a_2$.*

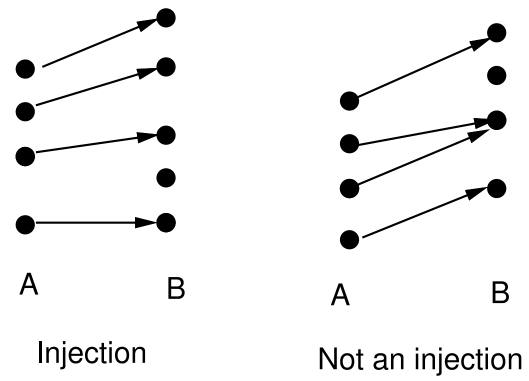
In an injection, no two elements in A point to the same element.

Definition 1.4 *A function $f : A \to B$ is a* bijection *if it is both an injection and a surjection.*

In a bijection, there is a one-to-one correspondence between elements in A and elements in B.

For example, $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = 2x$ is one-to-one, but not onto, and thus not a bijection. However, if we modify the range, so $f : \mathbb{N} \to 2\mathbb{N}$("even numbers"), then we have a bijection.

If there is a bijection between two sets, then the two sets have the same size.



Theorem 1.5 *If A and B are finite sets and $f : A \to B$ is a bijection, then $|A| = |B|$.*

The idea is intuitive from the picture of a bijection. However, picture proofs can often be wrong. So, let's do a more formal proof.

Proof. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ where $n = |A|$. where the a_i are distinct. We actually haven't proved you can write the set this way, but we won't go into that in this class.

Claim: $B = \{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$

We need to show these two sets are equal. We do this by showing they are subsets of each other. Let $b \in B$. Since f is a bijection, there exists $a_i \in A$ such that $f(a_i) = b$. Thus, $B \subseteq \{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$. By definition of a function, each a_i maps into an element of B. Therefore, $B \supseteq \{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$. We have shown they are subsets of each other, thus $B = \{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$.

Also, note the $f(a_i)$ are distinct, since f is a bijection (hence, an injection). Therefore, $|B| = n$. ■

The converse of the theorem is also true. If two finite sets have the same size, then there is a bijection between the two. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$, then let $f(a_i) = b_i$ and we have a bijection. Thus, bijections show that two sets have the same size (cardinality).

Formally, cardinality is defined by a bijection.

Definition 1.6 *A set A has cardinality n, $|A| = n$, if there exists a bijection $f : A \to \{1, 2, 3, \ldots, n\}$.*

A seemingly immediate result is that any set A with more items than another set B cannot have a bijection between the two. This is true for finite sets. However, it is not true for *infinite* sets.

Consider $f : \mathbb{N} \to \{0, 2, 4, 6, 8, \ldots\}$ where $f(n) = 2n$. This is a bijection, yet the natural numbers contain more elements than the evens. Infinite sets have special properities. In the next lecture, we will discuss them further.

2 Pigeonhole principle

The principle can be simply stated as: "If $n+1$ pigeons are put in n holes, then at least one hole has two or more pigeons." One easy way to see this is that if each hole only had one or zero pigeons, then there are at most n pigeons. This simple fact allows us to prove many things. An immediate result is that if three people are in a room, then at least two have the same sex.

Here's a slightly more complicated example.

Theorem 2.1 *$\forall m < n$, every set with n numbers must contain two numbers whose difference is a multiple of m. (e.g., every set with 1000 numbers must contain two whose difference is a multiple of 573.)*

Proof. The proof is easy once you see how to apply the pigeonhole principle.

Let the n numbers be the pigeons. The holes are congruencies mod m (i.e. 0 mod m, 1 mod m, 2 mod m, ... , $m-1$ mod m). Thus, we can map pigeons into holes. Every pigeon will get mapped to some hole. Since there are more pigeons than holes, two pigeons get mapped to the same hole.

If two pigeons (say x and y) are in the same hole, they are congruent mod m. Thus, $x \equiv y (\mod m) \implies m | x - y$. ∎

Another example is the following.

Theorem 2.2 *Let "friendship" be a reflexive and symmetric relation. Then, any set of n people has two people with the same number of friends.*

Proof. We again use the pigeonhole principle. Let the pigeons be the n people. Now, we want to choose the holes so that there are $n-1$ holes, and every person gets mapped to a hole. The number of possible friends is 1 to n, which is n possibilites. Apparently, we can't directly apply the pigeonhole principle. However, if we do a little extra work, we can. If a person has n friends, then every person has at least two friends – himself and the very friendly person. Thus, the set of possible friendships is either $[1, n-1]$ or $[2, n]$. Each of these sets has $n-1$ elements, so we can apply the pigeonhole in either situation. Therefore, two people must have the same number of friends. ∎

The *Strong Pigeonhole Principle* is as follows. If $n(r-1)+1$ pigeons are put in n holes, then there is at least one hole with r pigeons.

We can prove this by contradiction. Assume all holes have $r-1$ or fewer pigeons, Then, the total number of pigeons is at most $n(r-1)$, which is a contradiction. Note the simple pigeonhole priciple is a special case when $r = 2$.

We can use the strong pigeonhole principle to prove some interesting facts. The expression "you're one in a million" is sometimes used to say a person is very special. Let's take its meaning literally. There are exactly one million different personalities any person can have. Beyond that, the differences are insignificant. In the country of China, there are well over a billion people. Thus, the strong pigeonhole principle implies there is one personality where at least 1000 people have that same personality. I guess those people aren't so special after all.