Massachusetts Institute of Technology | Handout 7
6.042J/18.062J: Mathematics for Computer Science | September 12, 1995
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Problem Set 2

Due: Start of Class, September 19.

Reading: Sections 3.1 to 3.3 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 2-1. Show that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ whenever n is a positive integer.

Problem 2-2. Give a recursive definition of the multiplication function $m \times n = mn$ in terms of the addition function $+$. Use your definition and the laws of addition to prove by induction that $m \times (n + p) = (m \times n) + (m \times p)$.

Problem 2-3. A *full binary tree* consists of either a *leaf node* with no children or an *internal node* with left and right subtrees that are full binary trees. Prove that a full binary tree on n nodes has $(n+1)/2$ leaves.

An example of a full binary tree is provided below:

Problem 2-4. n football teams participate in a *tournament*. In a tournament every pair of teams plays against each other, once. There are no ties. A *chain* is an ordering of the teams so that each team beats the one directly following it in the order. Prove that every tournament has a chain. (It may be useful to prove the following lemma as a step in the overall proof: in any sequence of n zeroes and ones that begins with a zero and ends with a one, there is a zero immediately preceding a one somewhere in the sequence.

Does this mean the first team in the order always beat all the other teams? Prove or give a counterexample.

Problem 2-5. Find the fallacies in the "proofs" of the following "theorems:"

(a) *Theorem:* Whenever a is a nonzero real number, we have $a^n = 1$ for all natural numbers n.
BASIS STEP: $a^0 = 1$ is true by the definition of a^0.
INDUCTIVE STEP: Assume that $a^k = 1$ for all nonnegative integers k with $k \leq n$. Then note that

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

(b) *Theorem* Every natural number is either a prime or a perfect square.
BASIS STEP: 1 is a perfect square.
INDUCTIVE STEP Assume the theorem true for all $k < n$ and show it for n. If n is prime, we are done. Otherwise, since n is composite, it can be factored as rs for $r, s < n$. By inductive hypothesis, r and s are squares, say $r = u^2$ and $s = v^2$. Then

$$n = rs = (u^2)(v^2) = (uv)^2$$

showing that n is a perfect square.

(c) *Theorem:* Every positive integer can be described in fewer than fourteen words.
Proof: Let S be the set of all numbers that cannot be so described. It contains a least element. This element is "the smallest number that cannot be described in less than fourteen English words." But this is a thirteen-word description, a contradiction. Thus S must be empty, implying the theorem.

Problem 2-6. Prove that every third Fibonacci number is even. ($F_0 = 1, F_1 = 1, F_{i+2} = F_{i+1} + F_i, \forall i \geq 0$. The series of third Fibonacci numbers is $F_2, F_5, \ldots$.)