Problem Set 1

Due: Start of Class, September 12.

Reading: Sections 1.1 to 1.3 and 3.1 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 1-1. Let p, q, and r be the following propositions:

p : You get an A on the final exam.
q : You do every exercise in this book.
r : You get an A in this class.

Write the following propositions using p, q, and r and logical connectives (e.g. $\vee$, $\wedge$, $\neg$, $\Rightarrow$).

(a) To get an A in this class, it is necessary for you to get an A on the final.

(b) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

(c) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

(d) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Problem 1-2. Find a compound proposition involving the propositions p, q, and r that is true when exactly two of p, q, and r are true. The compound propositon is false otherwise.

Problem 1-3. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of the following sentences in terms of $P(x), Q(x)$, quantifiers, and logical connectives. For the universe of discourse for quantifiers use the set of all students in your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn't know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++.

e) No Russian speaking student knows C++.

Problem 1-4. Let $L(x, y)$ be the statement "x loves y," where the universe of discourse for both x and y is the set of all people in the world. Use quantifiers to express each of the following statements:

a) Everybody loves somebody.

b) There is somebody whom everybody loves.

c) Nobody loves everybody.

d) There is exactly one person whom everybody loves.

e) There are exactly two people whom Lynn loves

f) There is someone who loves no one besides himself or herself.

Problem 1-5. Let $P(x), Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of the following statements using quantifiers; logical connectives; and $P(x), Q(x)$, and $R(x)$, where the universe of discourse is the set of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

d) Does c) follow from a) and b)? If not, is there a correct conclusion?

Problem 1-6. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$ are logically equivalent. (The new variable y is used to combine the quantifications correctly.)

Problem 1-7. Determine whether each of the following arguments is valid. If an argument is correct, what rule of inference (from Rosen, section 3.1) is being used? If it is not, what fallacy (from Rosen) occurs?

(a) *Theorem:* if $n^2 > 1$, then $n > 1$. *Proof:* If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$, then $n > 1$.

(b) *Theorem:* The number $\log_2 3$ is irrational. *Proof:* A number is irrational if it is not the ratio of two integers. Therefore, since $\log_2 3$ cannot be written in the form a/b where a and b are integers, it is irrational.

(c) *Theorem:* If $n^2 \leq 9$ then, $n \leq 3$. *Proof:* Because if n is a real number with $n > 3$, then $n^2 > 9$.

(d) *Theorem* If $n \leq 2$, then $n^2 \leq 4$. *Proof:* Because if n is a real number with $n > 2$, then $n^2 > 4$.

Problem 1-8. Prove that the sum of an irrational number and a rational number is irrational using a proof by contradiction.

Problem 1-9. Prove by cases that if x and y are real numbers, then $|x + y| \leq |x| + |y|$, (where $|x|$ denotes the absolute value of x, which is x when $x \geq 0$ and $-x$ when $x < 0$).

Problem 1-10. Consider an 8×8 checkerboard in which each square is colored red or black. Prove that regardless of how it is colored, there are two squares at distance exactly 4 apart that have the same color (distance here is measured in city-block style: squares that share an edge are at distance 1, squares that share only a corner are at distance 2; a "knight move" is distance 3).

Problem 1-11. Using the level of rigor applied in the section on Peano's Axioms, prove the *cancellation law*:

$$\forall m, n, p \in N, (m + p = n + p) \Rightarrow m = n$$

Problem 1-12. What is wrong with the following proof that $2 = 1$?

$$x^2 = \overbrace{x + x + \ldots + x}^{x}$$

Differentiating both sides with respect to x, we get

$$2x = \overbrace{1 + 1 + \ldots + 1}^{x} = x$$

Hence $2 = 1$.