Solutions for Tutorial 10

Problem 1. N people leave their hats in a coatroom. On the way out the person in charge of the coatroom hands their hats back to them in an arbitrary order. Use inclusion-exclusion to count the total number of ways in which the hats can be handed back to them such that no person gets his/her own hat back.

First, start with the number of ways of giving back all the hats: $n!$. Now, what we want to do is to subtract all the ways of getting exactly $i = 1, 2 \ldots n$ hats back correctly, but we can't easily determine these numbers easily, so let's subtract $\binom{n}{1} \cdot (n-1)!$ which is the number of ways to set one hat correctly in it's place times the number of ways to rearrange the rest of the hats.

$n! - \binom{n}{1} \cdot (n-1)!$

Now unfortunately we've overcounted the number of ways of giving back exactly 2, 3 ... n hats correctly. When we set one hat to be correct and rearranged the rest, we double count depending on which of the two hats were set correctly. For example, we set hat 1 correctly, but have a permutation where only hat 1 and 4 are correct. Later, we set hat 4 correctly, and again come across this permutation with hat 4 and 1 correct. We have overcounted. In fact, for any i hats set correctly, we overcounted i times–once for each of the times we chose one of the i hats to be correct and then rearranged the rest.

Well, to compensate for the doubled counting of exactly 2 correct, let's add $\binom{n}{2} \cdot (n-2)!$–the number of ways of choosing exactly two correct and rearranging the rest.

If we keep up this pattern we get:

$$n! - \binom{n}{1} \cdot (n-1)! + \binom{n}{2} \cdot (n-2)! + \ldots \binom{n}{n} \cdot (n-n)! = \sum_{i=0}^{n} \binom{n}{i} \cdot (n-i)!$$

Notice that for the ith term, we overcount exactly k hats placed correctly $\binom{k}{i}$ times for $k \geq i$.

If we look at how many times we add (or subtract) exactly k hats correctly, (including in our initial count of how many ways there are of returning all the hats ($n!$)) we get:

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} = (1-1)^k = 0$$

So we have summed up 0 instances of the total number of ways to return exactly $k \geq 1$ hats correctly. Which leaves us with all the hats returned incorrectly.

$$\sum_{i=0}^{n} \binom{n}{i} \cdot (n-i)!$$