Problems for Tutorial 12

Problem 1. You have found favor in the eyes of a king from the small, unknown country, called Xyzzy. The king likes to reward his subjects for their cleverness and intelligence, and so he has offered you the following opportunity:

You are given n gold coins and n silver coins, and two clay jars. You may place the coins in the jars in any way you like, and will be given the opportunity to take a coin from one of the jars. (You have an equal likelihood of choosing either jar, and then an equal likelihood of choosing any coin from that jar.) How should you place the coins so as to maximize the probability of getting a gold coin?

Problem 2. In another small, unknown country with unknown name an unknown king is known to have a known number n of bags of gold coins hidden away in his treasury. It is also known that each bag contains exactly n gold coins. A well-known craftsman has been commissioned by the king to create a jewel-encrusted sculpture. What the king doesn't know is that the craftsman is also a counterfeiter. Unfortunately for the counterfeiter, he doesn't know how to countefeit jewels or anything else that would be useful in faking part of his sculpture. He does, however, know that he will have access to the treasury to deposit his sculpture and that he will have the opportunity to replace as many of the king's gold coins with counterfeit coins as he likes. He also knows that before he can flee the (small, unknown) country, a well-known auditor will give the bags a check. The craftsman knows that the check will consist of choosing one coin uniformly at random from each of the bags and giving it to a well-known expert to check whether it is counterfeit. He also knows that each selected coin is checked carefully enough that if it is one of his counterfeits, he will get caught. He doesn't know what the best strategy is for distributing coins to minimize his chances of getting caught (crafstman, yes; counterfeiter, yes; mathematician, no), so he determines to replace exactly m coins from each bag with counterfeit coins. He does not yet know what value of m he will use.

(a) For each value of m, what is the probability that he will get caught?

(b) Give an expression for what your answer above approaches as n approaches infinity.

(c) Use the expression from the above part to compute the craftsman's expected ill-gotten gain. What value of m maximizes this?

Problem 3. Statistical Mechanics

Statistical Mechanics is the branch of Physics that studies the statistical behavior of large collections of particles (e.g a gas in a container, an electron cloud in a conductor, radiation in a room). It turns out that "large" particles obey different statistics than particles that lie in the quantum mechanical regime.

In the classical theory developed by Maxwell and Boltzmann, particles are treated as distinguishable balls. They are considered distinguishable because they obey Newtonian mechanics therefore in principle they could be observed and labeled. Consider a collection of r distinguishable particles, and divide the container in which they lie into n cells [1]. Then the classical statistical mechanics problem becomes a balls and bins problem.

1. What is the total number of arranging the particles in the cells?
   answer :n^r
2. What is the probability p_k, of having exactly k particles in a particular cell?
   answer $p_k = \binom{r}{k}(\frac{1}{n})^k(1 - \frac{1}{n})^k$
3. Calculate the expected number of empty cells.
   Answer np_0
4. Calculate the average number of empty cells.
   answer p_0
5. Let $\lambda = \frac{r}{n}$, the average number of particles/cell, be constant, while $n \to \infty$ and $r \to \infty$. Show that the average number of empty cells goes to $e^{-\lambda}$.

In the quantum mechanical regime particles can not be treated as distinguishable entities. The explanation is involved and relies on the Heisenberg uncertainty principle [2]. The statistics derived by the assumption that particles are indistinguishable are called Bose-Einstein statistics. Now consider r indistinguishable particles in n cells.

1. What is the total number of arranging the particles in the cells?
   answer :$\binom{n+r-1}{r}$
2. What is the probability q_k, of having exactly k particles in a particular cell?
   answer $q_k = \frac{\binom{n+r-k-2}{r-k}}{\binom{n+r-1}{r}}$

[1] We formulate the problem in this way for visualization purposes. In statistical mechanics what gets divided into cells is not the volume, but the state space, i.e. the space defined by the positions and momenta of the particles. Correspondingly, we don't consider r particles, but rather r points in the state space

[2] The Heisenberg uncertainty principle states that the product of uncertainties in position and momentum always exceeds some constant $\Delta x \Delta p \geq \frac{\hbar}{2}$. If the size of the cells in the state space is of that order, then the positions and momenta of particles lying in the same cell are "blurred" by the uncertainty principle and they become indistinguishable

3. Calculate the expected number of empty cells.

   answer nq_0

4. Calculate the average number of empty cells.

   answer q_0

5. Let $\lambda = \frac{r}{n}$, the average number of particles/cell, be constant, while $n \to \infty$ and $r \to \infty$. Show that the average number of empty cells goes to $\frac{1}{1+\lambda}$.

Observe that when λ is large the two expectations are quite different, while when λ is very small (corresponding to either a large number of cells, i.e. an almost free system, or a small number of particles , i.e. a weekly interacting system) then the two quantities become identical. What are they equal to if second order terms are ignored?

answer $1 - \lambda$

In order to make our exposition complete, let us mention that there is a third kind of statistics, the Fermi-Dirac statistics, that applies to particles known as Fermions (electrons for example are Fermions). In this case the cells in state space correspond to different quantum states of the system. The Pauly exclusion principle states that no more than one Fermion can occupy the same quantum state, therefore this is a case where we can have at most one ball per bin. As a consequence there are only $\binom{n}{r}$ possible configurations.