Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Tutorial 11
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Problems for Tutorial 11

Problem 1. Warmup

Let $A_1, \ldots, A_n$ be a collection of disjoint sets such that their union is equal to the sample space. The following is true:

$P(B) = P(B \cap A_1) + \ldots + P(B \cap A_n)$

Use this to show that if $A_1, \ldots, A_n$ is a disjoint collection that covers the sample space, then for any events B, C such that $P(C) \neq 0$

$P(B \mid C) = P(B \cap A_1 \mid C) + ... + P(B \cap A_n \mid C)$

Observe the useful corollary:

$P(B \mid C) = P(B \cap A \mid C) + P(B \cap \bar{A} \mid C)$

Problem 2. In *Polya's Urn Model* we begin with an urn having b black and r red balls. Let $c > 0$ be an arbitrary natural number. At each time t ($t = 1, 2, \cdots$), we draw one ball from the urn, and put $c + 1$ balls of the same color into the urn. (So there are $b + r + c$ balls in the urn afterwards).

Show that the probability of drawing a sequence of t balls, so that the last ball is black is $b/(b + r)$.

(Hint: argue that the probability of a sequence is independent of the ordering. $P(BRB) = P(RBB)$, etc. Then the probability that a sequence starts with B is the same as the probability that the sequence ends with B for all sequences of length t.)

Problem 3. Mathematicians

(a) Suppose that A and C are to fight in a pistol duel. We know that A's chance of hitting his target is 0.3 and C's is 0.5. Assuming that C shoots first, what's the probability that A survives?

Now suppose that B enters the fray and A, B, and C participate in a three-cornered duel. A's and C's chances of hitting are same as before while B never misses. They are to fire at their choice of target (assume B and C shoot at the more "dangerous" adversary if there is a choice) in succession in the order A, B, C cyclically until one man is left standing.

(b) What's the probability that A survives if he targets B first?

(c) What's the probability that A survives if he targets C first?

(d) If A (and A alone) has the option of firing into the air (targeting no one) on the first round, what's the probability that he survives?

(e) What should A's strategy be in order to maximize his chances of surviving?

Problem 4. EXTRA!

The IRS has decided to audit k tax forms among those submitted. It doesn't know how exactly how many forms have been submitted (although it has many more than k). It will will make its selection in a one-time pass in chronological order over the received tax forms. Since it does not want to provide any incentive for taxpayers to file their forms at particular times, it wants each tax form to be equally likely to be audited.

It begins by tentatively selecting the first k forms received. As it visits each subsequent form, it has the option to replace one of the k currently selected forms by the latest one, so that it always has a selection of exactly k forms.

Devise a randomized strategy for the IRS, so that at any time during the one-time pass, every tax return visited to that point in the pass is equally likely to be in the current audit list.

Hint: Consider $k = 1$, and then try to generalize.