Problems for Tutorial 8

Problem 1. Lost In the Forest

Consider a forest with k connected components and n vertices. How many edges are in the forest?

Lemma: any tree with n vertices has $n-1$ edges:

Base case: 1 vertex, no edges

Inductive hypothesis: P(n): for any tree of size $k \leq n$ there are $k-1$ edges.

Assume $P(n)$, show $P(n+1)$:

Given a tree with $n+1$ vertices, take away one edge. We then have two subtrees of size n_a and n_b with $n_a + n_b = n+1$ and $n_a, n_b \leq n$. By the inductive hypothesis, the subtrees have $n_a - 1$ and $n_b - 1$ edges each. Since we took away one edge to form the subtrees, the total number of edges for a tree of size $n+1$ is $n_a - 1 + n_b - 1 + 1 = (n+1) - 1$.

Problem 2. Hard to Break

A 2-connected graph *is a graph in which 2 edges must be removed to disconnect it. What is the minimum number of edges that a 2-connected graph of n vertices must have?*

For a graph to be 2-connected, each vertex must have at least degree 2. So the minimum number of edges is $|E| = \sum_{i=1}^{n}(\frac{1}{2}\text{degree of vertex } i) = 2n/2 = n$.

Problem 3. Tournaments

A Tournament Graph *is a directed graph such that if u and v are vertices then either (u, v) is an edge or (v, u) is an edge but not both. A* Directed Hamiltonian Path *is a simple path in a directed graph that contains every vertex exactly once. Prove by induction on the number of vertices that every tournament graph has a directed Hamiltonian path.*

Proof by induction on the number of vertices:

Inductive Hypothesis: A Tournament graph of size n has a Directed Hamiltonian Path (DHP).

Base case: 2 vertices
Since for every two vertices, there is is one edge between them, then there must be a DHP in the direction of that edge.

Assume the hypothesis is true for a graph with n vertices, prove that it's true for a graph of size $n+1$ vertices.

Let $G = (V, E), |V| = n + 1$ be our tournament graph. Create a new graph, G' by removing the vertex v from G. So $G' = (V', E')$ where $V' = V - \{v\}$ and $E' = \{$ all edges in E except those containing $v\}$. Note that G' is still a tournament graph since there are still edges going between all the remaining nodes. Since G' has n nodes, by the inductive hypothesis it has a DHP.

Assign numbers to the vertices according to their order in the path, starting with v_1 and ending with v_n.

Now, add v to the DHP. There are three cases:

- Case 1: $(v, v_1) \in E$
  Add v to the beginning of the DHP.
- Case 2: $(v_n, v) \in E$
  Add v to the end of the DHP.
- Case 3: $(v_1, v) \in E$ and $(v, v_n) \in E$

  Let $S = \{v_i \mid (v, v_i) \in E\}$. We know that S is not empty because $(v, v_n) \in E$, so by well-ordering there must be a minimum element, $v_k \in S$. Since $v_1 \notin S$, $(v_{k-1}, v) \in E$. Therefore we can insert vertex v between vertices v_{k-1} and v_k thus creating a DHP of size $n + 1$.