Problems Solutions for Tutorial 6

1 Warmups

(a) Determine whether each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, is an injection, surjection or a bijection. Also, give the inverse if it exists.

- $f(x) = e^x$
- $f(x) = x^2 + 3$
- $f(x) = (x^2 + 1)/(x^2 + 2)$

(b) What is the only function on a set A that is also an equivalence relation?

Recall that a function is an injection if $f(x) = f(y) \to x = y$ and a surjection if $\forall y \in \mathbb{R} \exists x (fx) = y$. A function is a bijection if it is a surjection and an injection and the inverse only exists if it's a bijection.

(c)

- $f(x) = e^x$
  injection? yes if $e^x = e^y \to \ln e^x = \ln e^y \to x = y$
  surjection? No because e^x is never negative.
  bijection? no.
  No inverse
- $f(x) = x^2 + 3$ injection? no because $f(x) = f(-x)$ even though $x \neq -x$
  surjection? No because $x^2 + 3$ is never negative.
  bijection? no.
  No inverse.
- $f(x) = (x^2 + 1)/(x^2 + 2)$
  injection? no because $f(x) = f(-x)$ even though $x \neq -x$
  surjection? No because $(x^2 + 1)/(x^2 + 2)$ is never negative.
  bijection? no.
  No inverse.

(d) $f(x) = x$

To show a function is an equivalence relation, we need to show that it's reflexive, symmetric, and transitive. Pairs in this relation are $(x, f(x)) \in R$.

R is reflexive because $(x, f(x)) \in R \rightarrow (x, x) \in R$.

R is symmetric because $(x, f(x)) \in R \rightarrow (x, x) \in R \rightarrow (f(x), x) \in R$

R is transitive because

$$
\begin{aligned}
(x, f(x)) \in R \wedge (f(x), f(f(x))) \in R & \\
\rightarrow & \quad (x, x) \in R \wedge (x, f(x)) \in R \\
\rightarrow & \quad (x, x) \in R \wedge (x, x) \in R \\
\rightarrow & \quad (x, x) \in R \\
\rightarrow & \quad (f(x), x) \in R \\
\rightarrow & \quad (f(f(x)), f(x)) \in R
\end{aligned}
$$

2 Seven Pigeons

Let S be the set of numbers $7, 77, 777, \ldots$. Prove that there is an element of S that is divisible by 1997.

Let the pigeons be elements $s \mod 1997$ where $s \in S$, and the holes be the numbers $0 \ldots 1996$.

Since there are more than 1997 elements of S, some hoel must have two numbers, say s_i and s_j, with i and j 7's respectively. Without loss of generality, assume that $i < j$.

$s_j - s_i = 77 \ldots 700 \ldots 0$ where there are $j - i$ 7's and i 0's. So, $s_j - s_i = s_{j-i} \cdot 10^i$.

Since $s_j \mod 1997 = r = s_i \mod 1997$, $s_j - s_i \mod 1997 = 0$, so $1997 \mid (s_j - s_i)$. Since 1997 doesn't divide 10^i, it must divide s_{j-i}, so $1997 \mid s_{j-i}$..

3 Uncomputable Functions

A computable function from the naturals to the naturals is a finite piece of code (in your favourite programming language). The code takes as input a natural number and outputs a natural number. Prove that not all functions f from the naturals to the naturals are computable.

Recall that a function is a relation which is a set of ordered pairs. Normally when we think of functions, we think of things like $f(x) = x^2$ which have a pattern to the pairs, but we could just as easily randomly assign $f(i)$ for each $(i, f(i))$. In order to do this, we'd have to explicitly enumerate the pairs. Since there are an infinite number of natural numbers, would require infinite space.

4 The Power of Power Sets

Solutions to this will be available later (I'll mail people to let you know when they're done.)

Let G be the set of all functions $f : \mathbb{N} \to \{0, 1\}$.

(e) Prove that $|G| = |2^{\mathbb{N}}|$.

(f) Let U be an uncountable set and C any countable subset of U. Prove that $U - C$ is uncountable.

(g) Prove that $|2^{\mathbb{N}}| = |(0, 1)|$ and conclude that $|2^{\mathbb{N}}| = |\mathbb{R}|$.