Problems for Tutorial 2

1 Warmup

Prove by induction on k that

$$5 \mid (3^{2k} + 4^{k+1})$$

for any integer $k \geq 0$.

Inductive Hypothesis(k): $5 \mid (3^{2k} + 4^{k+1})$.

BASE CASE: k=0
$3^{2\cdot 0} + 4^{0+1} = 1 + 4 = 5$ which is divisble by 5.

INDUCTIVE STEP: $H(k) \rightarrow H(k+1)$

By the inductive hypothesis, $5 \mid 3^{2k} + 4^{k+1}$, so $3^{2k} + 4^{k+1} = 5q$

$$\begin{array}{rcl} 3^{2(k+1)} + 4^{(k+1)+1} & = & 3^{2k+2} + 4^{k+1+1} \\ & = & 9 \cdot 3^{2k} + 4 \cdot 4^{k+1} \\ & = & 10 \cdot 3^{2k} + 5 \cdot 4^{k+1} - (3^{2k} + 4^{k+1}) \\ & = 5(2 \cdot 3^{2k} + 4^{k+1}) - 5q & \\ & = & 5(2 \cdot 3^{2k} + 4^{k+1} - q) \end{array}$$

So $5 \mid 3^{2(k+1)} + 4^{(k+1)+1}$.

2 Peano Strikes Back

Recall that addition was defined by:

D1) $\forall n \in \mathbb{N}, n + 0 = 0 + n = 0$

D2) $\forall n, m \in \mathbb{N}, n + m' = (n + m)'$

Prove by induction the cancelation law of addition.

$$\forall n, m, k \in \mathbb{N}, n + k = m + k \implies n = m$$

Induct over k
BASE CASE: $k = 0$
$\forall n, m \in \mathbb{N}, n + 0 = m + 0 \implies n = m$ by D1.

INDUCTIVE STEP: $H(k) \rightarrow H(k + 1)$
$\forall n, m \in \mathbb{N}, n + k' = m + k' \implies$
$\forall n, m \in \mathbb{N}, (n + k)' = (m + k)' \implies$ (by D2)
$\forall n, m \in \mathbb{N}, (n + k) = (m + k) \implies$ (by Peano's axioms)
$\forall n, m \in \mathbb{N}, n = m$ (by the inductive hypothesis).

3 Handshaking Problem

I recently went to a party with my spouse; there were 11 other married couples there. A lot of handshaking happened (but of course no one shook hands with their spouse). At the end, it turned out that for every k from 0 to 22, someone other than me had shaken hands with exactly k people. How many people did I shake hands with? Prove it.

Note that there are 24 people (12 couples).

Think of this problem as couples taking turns walking around and shaking people's hands, and then leaving the room. Number the couples 0 through 11.

Our inductive hypothesis, $H(n)$, is that when the nth couple shakes hands, one spouse shakes hands n times, and the other shakes hands $22 - n$ times. Furthermore, everyone who hasn't walked around yet has shaken hands $n + 1$ times.

BASE CASE: $n = 0$
If someone has shaken hands 22 times, they shook hands with everyone except themselves and their spouse. Since someone had to shake hands 0 times, their spouse had to have shaken hands with noone. Furthermore, everyone else has shaken hands once, since the first spouse shook hands with 22 different people.

INDUCTIVE STEP: $H(n) \rightarrow H(n + 1)$

When the next couple walks around to shake hands with everyone who's left, everyone has shaken hands $n + 1$ times by the inductive hypothesis. Since someone must shake hands exactly $n + 1$ times, (and since, by our hypothesis, we don't know who this is yet, but we do know who's shaken hands 0 thru n times), one of the spouses doesn't shake hands with anyone. Since someone must shake hands exactly $22 - n$ times, the other spouse, who has shaken hands n times already, must shake hands with all of the remaining $22-2n$ people for a total of $22-n$ times, otherwise no one will be able to shake hands $22-n$ times because there won't be enough people left. All the remaining people have now shaken hands one additional time, for a total of $n+2$ times.

This shows that $H(n) \rightarrow H(n+1)$, since one spouse shakes hands $22-n$ times, another n times, and the total number of handshakes done by everyone left is $n+2$.

Since we know that no one can shake hands more than 22 or less than 0 times, the speaker must shake hands the same number of times as someone else. Examining the inductive hypothesis, we realize that the extra person must shake hands 11 times, since two people shake hands 11 times (when $n = 11$).