Final

- This quiz is closed book. Three crib sheets are allowed.
- Put your name on the top of **every** page—**these pages will be separated for grading.**
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat**.
- This quiz ends at 12 noon. It contains 13 problems. You will get 10 extra points for writing your name, tutorial TA and time clearly.
- GOOD LUCK!

Problem	Points	Grade
name	10	
1	15	
2	15	
3	10	
4	10	
5	10	
6	15	
7	15	
8	15	
9	15	
10	20	
11	15	
12	15	
13	20	
Total	200	

Name: ______________________________

Please circle your tutorial TA: Terry David Dimitri

Please circle your tutorial time: 1 2 3 7

Problem 1. [15 points] Circle True/False.

T F $p \wedge (\neg p \rightarrow (p \vee \neg q))$ is a tautology

T F $((\forall x)[P(x) \implies (\forall y)Q(y)]) \wedge (\exists z)\neg Q(z)$ implies $(\forall w)\neg P(w)$.

T F There is an integer solution to the equation $17x + 144y = 3$

T F The function $f : \mathbb{Q}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$ is a surjection

T F The function $f : \mathbb{Q}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$ is an injection

T F The relation $\{(A, B) \mid$ there is a surjection from A to $B\}$ is transitive.

T F The relation $\{(f, g) \mid f = \Theta(g)\}$ on functions is an equivalence relation.

T F If R and S are transitive relations on A, then $R \circ S$ is transitive.

T F If R and S are reflexive relations on A, then $R \cap S$ is reflexive.

T F The set of prefixes of the decimal representation of π is uncountable.

T F The set of real numbers between 0 and 1 that have only 2s and 9s in their decimal representation is countable.

T F $|2^{\mathbb{R}}| < |\mathbb{R}^5|$

T F There is no set whose power set has the same cardinality as the natural numbers.

T F A 14-vertex graph with simple cycles of 10 different lengths is not bipartite.

T F $\log(n!) = o(n \log n)$

Problem 2. [15 points] Short answers. No need to be rigorous, though partial credit may be given based on the derivation.

(a) What equivalence relation on a nonempty set A is also a poset on A?

(b) How many simple cycles of length 7 are there in an n vertex complete graph?

(c) What is the coefficient of x^{30} in $(5+x)^{50}$?

(d) The space shuttle runs 3 identical computers so that if one goofs, the other two will outvote it with the right answer. If each computer goofs independently with probability 10^{-7}, what is the probability that the right answer will prevail?

(e) Suppose I flip 6 coins and then roll a 6-sided die for each head I see. What is the expected number of spots I will see?

(f) Give asymptotic (Θ) bounds for the summation

$$\sum_{k=1}^{n} \frac{\ln k}{k}$$

Problem 3. [10 points] Consider the following **fallacious** claim and proof.

Claim: if every horse in a herd is the same color as some other horse in the herd, then all horses in the herd are the same color.

Proof: Induction on the number of horses. Base case of 2 horses trivial: under the hypothesis, one must be the same color as the other. So consider a set of n horses satisfying the criteria, and suppose we add an $(n+1)^{st}$ horse. By assumption, this horse is the same color as some other horse. By inductive hypothesis, that horse is the same color as all the other horses. So all the horses including the $(n+1)^{st}$ are the same color.

(a) Give a counterexample to the claim.

(b) Explain **succinctly** why the proof is wrong.

Problem 4. [10 points] Prove that $(A - B) \bigcap (A \bigcup \overline{B}) = A \bigcap \overline{B}$

Problem 5. [10 points] Suppose that there are 6 possible mistakes that could be made in answering the previous question. Given that the class has 120 students, show that two of them will make exactly the same set of mistakes.

Problem 6. [15 points] Consider the following algorithm on a poset $\preceq$ over $x_0, \ldots, x_n$.

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x ← x_0
for k = 1 to n
    if x_k ⪯ x then
        x ← x_k
return x
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Prove that the above algorithm returns a minimal element of the poset.

Problem 7. [15 points] Let $G = (N, E)$ be a simple graph, i.e., an undirected graph with no self-loops. A subset of nodes $S \subseteq N$ is an *independent set* if no two nodes in the set are adjacent (connected by an edge). Prove **by induction on the number of vertices** that any graph with maximum degree Δ has an independent set of size at least $n/(\Delta + 1)$ (Hint: consider removing a vertex and its neighbors).

Problem 8. [15 points] Solve the following recurrences (show your derivation):

(a) The recurrence

$$T(n) = 2T(n/2) + 6n$$

subject to the initial condition $T(1) = 1$.

(b) The recurrence

$$f(n) = 4f(n-1) - 4f(n-2)$$

subject to the initial conditions $f(0) = 0$, $f(1) = 3$.

(c) As I go down a flight of stairs, each time I take a step I can choose to descend 1 stair or 2 stairs. Derive (and justify) a recurrence (and base cases) for the number of different ways I can go down n stairs (two ways are different if they encounter different sets of steps). What numbers does this recurrence define?

Problem 9. [15 points] Consider the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

(a) Prove this identity using the binomial theorem.

(b) Prove this identity via a combinatorial argument. (Hint: consider the number of ways to donate some subset of your n baseball cards to two museums. For one side, consider where each card ends up. For the other, consider where the cards you donate end up)

Problem 10. [20 points] My favorite bagel store offers 14 different kinds of bagels and 30 different spreads.

(a) How many different items does the store sell?

(b) How many different sandwiches are available (1 bagel variety, 1 spread)?

(c) 150 bagels in the store have poppy seeds on them, 200 have sesame seeds, and 100 have rye seeds. Among those bagels, 100 have sesame and poppy, 80 have sesame and rye, and 40 have poppy and rye. Among those, 20 have all three kinds of seeds. How many bagels have at least one kind of seed?

(d) How many distinguishable bags of 12 bagels (no spreads) can I buy? (Two bags are indistinguishable if they both have the same number of each bagel variety.)

(e) If I choose one of those distinguishable bags uniformly at random, what is the probability that I get 12 different bagel varieties?

(f) If I choose a spread for each of my 12 bagels uniformly at random, what is the probability that all of my bagels get different spreads?

(g) Suppose I throw each of my 12 bagels randomly into one of 3 bags. What is the probability that I get 4 bagels in each bag?

Problem 11. [15 points] 20% of the theorems in the lecture notes contain typos. For any theorem, it is the case that if you ask a TA whether it has a typo then the probability that their answer is correct is 0.9. TAs answer independently of one another.

(a) Given a theorem selected uniformly at random, if a TA tells you that it has a typo, what is the probability that it does?

(b) Suppose you pick a random theorem, ask 1 TA if there is a typo, and then ask a second TA is there is a typo. Draw the sample space of outcomes of this experiment.

(c) If one TA tells you a theorem has a typo, what is the probability a different TA will tell you the same theorem has a typo?

Problem 12. [15 points] I have an urn containing balls of c distinct colors, where there are k_i balls of color i and $\sum k_i = n$. I perform the following experiment:

1. pick a ball, note its color, and put it back
2. repeatedly choose a ball from the urn and replace it, until I get a ball the same color as the first.

(a) Conditioning on the fact that the first ball has color i, what is the expected number of picks I make in the second step?

(b) If I do not condition on the color of the first ball, what is the expected number of picks I make in the second step (simplify your answer)?

Problem 13. [20 points] When we grade this exam, our plan is the following. Independently for each of the 14 whole questions (**including your "name"**), we will randomly assign full credit (with probability 3/5) or no credit (with probability 2/5).

(a) What is your expected score on problem 1?

(b) What is your expected score on the exam?

(c) Use the expectation to bound the probability that your score exceeds 160?

(d) What is the variance of your score on problem 1?

(e) What is the variance of your exam score?

(f) Use the variance to bound the probability that your score is less than 90?

(g) Prove that the probability you get a 0 (that is, that no credit is assigned to any one of the 14 problems) is less then $1/100$.

(h) Suppose that instead of making an independent random choice for each question, we assigned full credit on all of them with probability $3/5$ and 0 credit on all of them otherwise. Which of the previous problem parts have unchanged answers, and why?

(i) Suppose instead that the exam consisted of 10000 true/false questions, and that we randomly chose to give you credit for each independently with probability $3/4$. Show that the probability that you get more than 8000 credits is less than e^{-10}. You may use the fact that $\ln(20/19) + 19/20 - 1 > 0.00129$.