Problem Set 2

Due: Start of Tutorial, Tuesday, February 18.

Problem 1. Prove by induction that $n^5 - n$ is a multiple of 5 for every integer n.

Problem 2. Determine and prove which Fibonacci numbers are divisible by 3.

Problem 3. Find a formula for

$$\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n. Use mathematical induction to prove your result.

Problem 4. Use mathematical induction to prove that $\forall n \geq 4$, $n^2 < n!$.

Problem 5. Define a new propositional operator $\oplus$ as follows: for propositions p and q, $p \oplus q$ is true if exactly one of p and q is true.

(a) Write $p \oplus q$ in terms of the standard operators $\vee, \wedge, \neg$.

(b) Let $P_1, \ldots, P_n$ be n propositions and Q_n the proposition $P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Prove by induction on n that Q_n is true if and only if the number of true propositions among $P_1, \ldots, P_n$ is odd.

Problem 6. What are the fallacies in the following arguments? Don't just explain why the claim is wrong; rather, explain how the argument violates the notion of correct proofs.

(a) Claim: Every integer is nonnegative. Proof: 0 is nonnegative. Assuming n is nonnegative, $n + 1$ is certainly nonnegative.

(b) Claim: Every natural number is a prime or a perfect square. Proof: as the inductive hypothesis, use $P(n) =$"every number less than or equal to n is a prime or a perfect square." $P(1)$ is certainly true. Now consider n. If n is prime we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n - 1$. By the inductive hypothesis, r and s are perfect squares, $r = u^2$ and $v = s^2$. Then $n = rs = u^2v^2 = (uv)^2$.

(c) Claim: no quantity of homework problems is too many. Proof: Base case: Clearly 0 homework problems is not too many. Inductive step: if n homework problems are not too many, then certainly $n + 1$ homework problems is not too many.

Problem 7. Using only the Peano Axioms, prove

$$\neg(\exists x, y)(x = y' \wedge y = x')$$

(recall that x' is the successor of x).

Problem 8. Give a recursive definition of multiplication of natural numbers in terms of addition. Using standard facts about addition (associativity, commutativity, etc) as axioms, prove that $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

Problem 9. **(Optional).** A disgraceful disease strikes a monastery. The monks are informed that at least one of them has the disease. All monks are ordered to leave the monastery as soon that they are certain that they are infected. The disease demonstrates itself by a mark on the forehead (all sick people get the mark). Unfortunately in this monastery there are no mirrors and the monks are not allowed any form of communication. Monks gather to pray every morning so they can all see each other. If they become certain that they are infected they depart during that night.

Use an inductive argument to show that if the monks are rational and there are n **infected** monks then they will all depart the monastery by the n^{th} day.