Solutions to Quiz 1

- This quiz ends at 9 P.M. It contains 10 problems. You have 120 minutes to earn 100 points.
- This quiz is closed book. You may use one $8\frac{1}{2}'' \times 11''$ handwritten crib sheet.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

Problem	Points	Grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Name: ______________________________

Figure 1: Quiz 1 Statistics.

Problem 0-1. [10 points] Use the well-ordering principle to prove that $\sqrt[3]{2}$ is irrational.

Problem 0-2. [10 points] Use induction to prove that the sum of the first n odd numbers is n^2.

Problem 0-3. [10 points] Let R_1 and R_2 be binary relations on a set A. Prove that if R_1 and R_2 are symmetric, then $R_1 \circ R_2$ is symmetric, or provide a counterexample.

Problem 0-4. [10 points] Use Stirling's formula $n! \sim \sqrt{2\pi n}\,(n/e)^n$ to show that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \; .$$

Problem 0-5. [10 points] Prove that in any undirected graph, the sum of the squares of the vertex degrees is even, or provide a counterexample.

Problem 0-6. [10 points] Consider a binary tree T, and let L be the set of leaf nodes of T. For any node x in T, define $w(x) = 2^{-d}$, where d is the depth (distance from the root) of x. Prove by induction that

$$\sum_{x \in L} w(x) \leq 1 \ .$$

Problem 0-7. [10 points] Consider the following recursive sorting algorithm:

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SORT(A, i, j)
if A[i] > A[j]
    then exchange A[i] ↔ A[j]
if i + 1 ≥ j
    then return
k ← ⌊(j − i + 1)/3⌋          ▷ Round down.
SORT(A, i, j − k)            ▷ Sort first two-thirds.
SORT(A, i + k, j)            ▷ Sort last two-thirds.
SORT(A, i, j − k)            ▷ Sort first two-thirds again.
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(a) Argue that $\text{SORT}(A, 1, n)$ correctly sorts the input array $A[1..n]$.

(b) Give a recurrence for the worst-case running time of SORT and a tight asymptotic (Θ) bound on the worst-case running time.

(c) Compare the worst-case running time of SORT with that of insertion sort and merge sort.

Problem 0-8. [10 points]

For each function $f(n)$ along the left side of the table below and each function $g(n)$ across the top, write O, Ω, or Θ in the appropriate space, depending on whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write only the strongest one. *Wrong answers will be penalized, so do not guess unless you are reasonably sure.* The first line is a demo solution.

	n	$n \lg n$	n^2
$3n \lg n$	Ω	Θ	O
$n^{0.99} \lg^2 n$			
$2n^2 - 5n \lg n$			
$\lg(n!)$			
$3^{\lg n}$			
$\sum_{i=1}^{n} \lg i$			

Problem 0-9. [10 points] Use the substitution method (guess and verify) to determine an asymptotic upper (big O) bound on the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n \leq 3, \\ T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + n & \text{if } n > 3. \end{cases}$$

Problem 0-10. [10 points] Use the method of characteristic equations to solve the linear recurrence

$$f(n) = 5f(n-1) - 6f(n-2) + 7^n ,$$

where $f(1) = 1$ and $f(2) = 2$. (For partial credit, solve only the homogeneous recurrence, that is, drop the 7^n term.)