Answers to practice problems for the final exam

These answers are provided for you to check your work. You should try each problem for at least 15 minutes before consulting these answers.

Problem 0-1. $6 \cdot 5 \cdot \binom{5}{3} \cdot \left(\frac{1}{6}\right)^5$.

Problem 0-2. **F, T, T, F, T, F, F, T, T.**

It is somewhat amazing that number 4 is false. It was intended to be true, but Ravi pointed out that a counterexample is $t(n) = 2^{1/n}$ and $p(n) = 2^{1-1/n}$.

Problem 0-3. Proof sketch: There are four $\frac{1}{2} \times \frac{1}{2}$-squares in the unit square so by the pigeonhole principle, some two points inside the same one, where they can be at most distance $\sqrt{(1/2)^2 + (1/2)^2}$ apart.

Problem 0-4. Proof sketch: Verify true for $n = 2$, then start with the inequality for $n-1$ and add $1/\sqrt{n}$ to both sides and notice that the right hand side is

$$\sqrt{n-1} + \frac{1}{\sqrt{n}} = \sqrt{n}\left(\sqrt{1 - \frac{1}{n}} + \frac{1}{n}\right) > \sqrt{n}.$$

Problem 0-5.

(a) $T(n) = \Theta(n^3)$

(b) $T(n) = \Theta(n)$

(c) $T(n) = \Theta(n^2 \lg n)$

(d) $T(n) = \Theta(n^2)$

(e) $T(n) = \Theta(n^{\log_2 7})$

(f) $T(n) = \Theta(\sqrt{n} \lg n)$

(g) $T(n) = \Theta(n^2)$

Problem 0-6. Proof sketch: Show $n = R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell)$ for $k, \ell \geq 1$. Note that if k or ℓ is ≤ 1, then $n = 1$ (the (trivial) graph on one node is sufficient). Take as the base case $k + \ell = 4$: for k or ℓ equal to 1, we have $1 \leq 1 + 1$; for $k = \ell = 2$, we have $n = 2$ (either the two nodes are connected or not) which is equal to $1 + 1$. For the inductive case, fix a node v. Either (1) v has edges to $R(k - 1, \ell)$ nodes, on which either there are already ℓ nodes with no edges between them or there are $k - 1$ nodes with all edges between them, which together with v makes k nodes with all edges between them, or (2) v has no edge to $R(k, \ell - 1)$ nodes, on which either there are already k nodes with all edges between them or there are $\ell - 1$ nodes with no edges between them, which together with v makes ℓ nodes with no edges between them.

Problem 0-7.

(a) $\sigma_{cR} = c\sigma_R = cs$.

(b) $\mathrm{Var}(R)$ may be arbitrarily large.

(c) $\mathrm{Ex}[1/R] \leq 1$.

(d) $\mathrm{Prob}[R = 10] = 3/4$ and $\mathrm{Prob}[R = 4] = \mathrm{Prob}[R = 16] = 1/8$.

Problem 0-8.

$$
\begin{aligned}
T(n) &= (n - \lg n) + \max\left(1 + T(\lg n), 1 + \lg n\right) \\
&\leq (n - \lg n + 1) + T(\lg n) \\
&\leq (n + 1) - \lg n + (\lg n - \lg\lg n + 1) + T(\lg\lg n) \\
&\leq (n + 2) - \lg\lg n + (\lg\lg n - \lg\lg\lg n + 1) + T(\lg\lg\lg n) \\
&\vdots \\
&\leq n + \lg^* n.
\end{aligned}
$$

(For the base case, notice either that $T(1) = 0 \leq 1 + 0$ or that $T(2) = 2 \leq 2 + 1$.)

Problem 0-9.

(a) $G_X(x) = \frac{1}{3}x + \frac{1}{6}x^3 + \frac{1}{2}x^4$.
$G_Y(x) = \frac{3}{4}x^2 + \frac{1}{4}x^5$.

(b) $G_R(x) = \frac{3}{12}x^3 + \frac{3}{24}x^5 + \left(\frac{3}{8} + \frac{1}{12}\right)x^6 + \frac{1}{24}x^8 + \frac{1}{8}x^9 = G_X(x)G_Y(y)$.

Problem 0-10.

(a) (This was proved in lecture. It requires that X and Y be independent.)

(b)

$$\begin{aligned}\mathrm{Ex}[R] &= G'(1)\\ &= G'_X(1)G_Y(1) + G_X(1)G'_Y(1)\\ &= G'_X(1) + G'_Y(1)\\ &= \mathrm{Ex}[X] + \mathrm{Ex}[Y].\end{aligned}$$

(c) (The rest is straightforward manipulation...)

Problem 0-11.

(a)

$$G_R(x) = \sum_{k\geq 1}(1-p)^{k-1}px^k = xp\sum_{k-1\geq 0}(1-p)^{k-1}x^{k-1} = \frac{px}{1-x(1-p)}.$$

(b) $G'_R(x) = p/[1-x(1-p)]^2$.
$G''_R(x) = 2p(1-p)/[1-x(1-p)]^3$.
$\mathrm{Var}(R) = 2(1-p)/p^2 + 1/p - (1/p)^2 = (1-p)/p^2$.

Problem 0-12.

(a) Sum of k balls $= \sum_{i=1}^{i=n} i \cdot B_i$.

(b)

$$\mathrm{Prob}[B_i = 1] = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

(c)

$$\mathrm{Ex}[\mathrm{Sum}] = \sum_i i\,\frac{k}{n} = k\,\frac{n(n+1)}{2n}.$$

Problem 0-13.

(a) Else

$$\mathrm{Ex}[R] < L\frac{\mathrm{Ex}[R]-k}{L-k} + k\left(1-\frac{\mathrm{Ex}[R]-k}{L-k}\right) = \frac{L\mathrm{Ex}[R]-k\mathrm{Ex}[R]}{L-k} = \mathrm{Ex}[R],$$

a contradiction.

(b) 1/6

(c) 0. (Could be he scores 50 with probability 1.)

(d) If $\mathrm{Prob}[R > k] \geq p$ then $\mathrm{Prob}[R \leq k] \leq 1 - p$.
So $\mathrm{Prob}[R < 0] \leq 1 - 1/4 = 3/4$.

(e) 1. $\mathrm{Prob}[R \leq 50] \leq 1$ is best bound for the same reason as part (c).
For any $p < 1$, it is possible that he scores $50 - \epsilon$ with probability p and 200 with probability $1 - p$, where $\epsilon = 150(1-p)/p$ to make $\mathrm{Ex}[R] = 50$.

(f) Thinking of everything complemented, we have

$$\mathrm{Prob}[R < k] = \mathrm{Prob}[-R > -k] \geq \frac{\mathrm{Ex}[-R]-(-k)}{50-(-k)}$$

since $-R \leq 50$ always. So $\mathrm{Prob}[R < 100] \geq (-50+100)/(50+100) = 2/3$.
(Else $\mathrm{Ex}[R] > (-50)\frac{1}{3} + 100\frac{2}{3} = 50$, a contradiction.)

(g) 1/4.

(h)

$$\mathrm{Prob}\left[(R-\mathrm{Ex}[R])^6 > 50^6\right] \leq \mathrm{Prob}\left[(R-\mathrm{Ex}[R])^6 > 5^6\cdot\mathrm{Ex}[(R-\mathrm{Ex}[R])^6]\right] \leq 1/5^6.$$

Problem 0-14. (Should have said, "using the calculation from Problem 11".)

$\mathrm{Var}(R_i) = (1-p)/p^2 = ni/(n-i)^2$.

$$\begin{aligned}
\mathrm{Var}(R) &= \sum_{i=0}^{n-1}\frac{ni}{(n-i)^2} = \sum_{i=0}^{n-1}\frac{n(n-i)}{i^2} \\
&= n\sum_{i=0}^{n-1}\frac{n-i}{i^2} \\
&= n\sum_{i=0}^{n-1}\left(\frac{n}{i^2}-\frac{1}{i}\right) \\
&\sim n\left(n\left[-\frac{1}{2i}\right]_1^{n-1} - \ln n\right) \\
&\sim \frac{n^2}{2}.
\end{aligned}$$

Problem 0-15. $\text{Ex}[return] = \sum_{k=0}^{k=\infty} \$2^k \left(\frac{1}{2}\right)^k$, which is infinite!
(You are still welcome to come play the game for \$1000 in NE43-371. . .)

Problem 0-16.
$\text{Ex}[\#heads] = 100\frac{1}{2} = 50.$
$\text{Var}(\#heads) = 100\frac{1}{2}(1 - \frac{1}{2}) = 25.$

(a) 5/7.

(b) 1/16.

(c)

$$\frac{1-0.3}{1-0.6} \cdot \sqrt{\frac{2}{100\pi}} \cdot e^{-4^2/2} \cdot \left(\frac{100}{e}\right)^{\frac{1}{1200}} \approx 0.000047$$

Problem 0-17. Prob[each unit does not fail] $= (1-p)^A \leq e^{-pA}$.