Some practice problems for the final exam

The final will contain problems that draw on material from the entire semester. The emphasis will be on probability and topics covered since the last exam. This handout contains some review problems and some practice problems on the material since the last problem set.

Problem 0-1. If you roll five fair, (mutually) independent dice, what is the probability of getting a full house (two dice with one value and three dice with a different value)?

Problem 0-2. True or false.
Let p, r, s and t be functions from $\mathbf{N}$ to $\mathbf{R}$.

- **T / F** $2^{\lg^2 n}$ is $\Theta(n^2)$.
- **T / F** $p(n) = 10(\frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^n})$ is $O(1)$.
- **T / F** If $t(n)$ is $\omega(1)$ and $p(n)$ is $O(1)$, then $t(n) + p(n) \sim t(n)$.
- **T / F** If $t(n)$ is $\Theta(p(n))$ then $\lg(t(n)p(n))$ is $\Theta(\lg t(n))$.
- **T / F** If $p(n) = 5$ then $p(n)$ is $O(1)$.
- **T / F** If $p(n) \sim s(n)$ and $t(n) = o(1)$, then $p(n) \sim s(n) + t(n)$.
- **T / F** If $p(n)$ is $O\left(s(n)t(n)\right)$ then $p(n)$ is $O(s(n))$ and $O(t(n))$.
- **T / F** $p(n) = 2n$ is a bijection between $\mathbf{N}$ and $2\mathbf{N}$ (the even natural numbers).
- **T / F** $p(n) = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}$ is $\sim \ln(\sqrt{n})$.

Problem 0-3. Suppose $p_1, p_2, \ldots, p_5$ are any five points in the interior of the unit square in the plane, and let d_{ij} denote the (Euclidean) distance between p_i and p_j. Prove that there exist p_i and p_j such that $d_{ij} < \sqrt{2}/2$. [Use the pigeonhole principle.]

Problem 0-4. Prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \qquad \text{for } n = 2, 3, 4, \ldots$$

Problem 0-5. Give asymptotic upper bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible.

(a) $T(n) = 2T(n/2) + n^3$

(b) $T(n) = T(9n/10) + n$

(c) $T(n) = 16T(n/4) + n^2$

(d) $T(n) = 7T(n/3) + n^2$

(e) $T(n) = 7T(n/2) + n^2$

(f) $T(n) = 2T(n/4) + \sqrt{n}$

(g) $T(n) = T(n-1) + n$

Problem 0-6. Recall that on Problem Set 2 you were asked to prove that any graph on 6 vertices contains either 3 vertices with no edges between them or 3 vertices with all edges between them. The idea was to fix a particular node v and look at its set of neighbors: if they number three or more, then either none of them share an edge (in which case we're done), or two of them share an edge, in which case they and v have all edges between them. If v's neighbors number 2 or fewer, a complementary argument is carried out on the nodes not adjacent to v.

This is a special case of what is called a "Ramsey Theorem". One Ramsey Theorem states that

> For any pair of positive integers (k, ℓ) there exists an n large enough so that *every* graph on n nodes has either a subset of k vertices with all $\binom{k}{2}$ edges between them or a subset of ℓ vertices with no edge between any two of them.

Prove this by induction on $k+\ell$ by assuming it is true for $(k-1, \ell)$ and $(k, \ell-1)$ and proving it's true for (k, ℓ). It may help to write n as a function of k and ℓ, $n = R(k, \ell)$, and show $R(k, \ell) \leq R(k, \ell-1) + R(k-1, \ell)$.

Problem 0-7. Let $R : S \to \mathbf{N}$ be a random variable.

(a) If $\sigma_R = s$ and c is a constant, what is σ_{cR}?

(b) If $\text{Ex}[R] = 1$, how large can $\text{Var}(R)$ be?

(c) If R is always positive (nonzero), how large can $\text{Ex}[1/R]$ be?

(d) Exhibit (sketch) a possible distribution for R with

- $\text{Ex}[R] = 10$,
- $\sigma_R^2 = 9$, and
- $\text{Prob}[|R - \text{Ex}[R]| \geq 6] = 1/4$.

Problem 0-8. Consider the following problem of communication complexity. You and I are each given a string of n bits—I get $x \in \{0,1\}^n$ and you get $y \in \{0,1\}^n$. Neither of us knows the other's string, but we know that they are different, $x \neq y$. We must determine a position in which our strings differ—a bit i, $1 \leq i \leq n$ such that $x_i \neq y_i$. We communicate by sending one-bit messages back and forth according to some algorithm, and we would like to minimize the total number of bits sent in the worst case. At the end of the algorithm, both of us must agree on a position in which our strings differ. For example, a simple algorithm would be for me to simply send my whole string, and for you to tell me the answer (i). This requires exactly $n + \lg n$ bits.

In fact, this problem can be solved with only $n + \lg^* n$ bits[1], using a clever recursive algorithm: I send you my first $n - \lg n$ bits, and either (1) you now know a bit in which our strings differ, in which case you first send me a 1 (to tell me this is the case) and then you send me the answer, or (2) our strings are the same in the first $n - \lg n$ bits, in which case you first send me a 0 (to tell me this is the case) and then we recurse on the remaining $\lg n$ bits.

Write a recurrence for $T(n)$, the total number of bits required by the algorithm. Note that in case (1) above, we finish with a total of only $(n - \lg n) + (1 + \lg n) = n + 1$ bits; you should convince yourself that it's the recursive case (2) that costs the most and therefore determines maximum number of bits required. Solve the recurrence by iterating it and discovering a telescoping sum.

Problem 0-9. Say random variable $X : S \to \mathbf{N}$ has distribution

$$X = \begin{cases} 1 & \text{with probability } 1/3 \\ 3 & \text{with probability } 1/6 \\ 4 & \text{with probability } 1/2 \end{cases}$$

and random variable $Y : S \to \mathbf{N}$ has distribution

$$Y = \begin{cases} 2 & \text{with probability } 3/4 \\ 5 & \text{with probability } 1/4 \end{cases}$$

and X and Y are independent.

(a) What is the probability generating function for X? For Y?

(b) Let random variable $R : S \to \mathbf{N}$ be given by $R = X + Y$.
Compute the probability generating function for R.
How is it related to the probability generating functions for X and Y?

[1] Recall that $\lg^* n$ is the least integer i such that $\lg^i n \leq 1$. Note that $\lg *n$ has a different "type" than $\lg^i n = \underbrace{\lg \lg \cdots \lg}_{i} n$ (undefined if $\lg^{i-1} \leq 0$); for $\lg^* n : \mathbf{R} \to \mathbf{N}$ but $\lg^i n : \mathbf{R} \to \mathbf{R}$. Also, $\lg^* n$ grows extremely slowly—$\lg^*(2^{65536}) = 5$—so $n + \lg^* n$ is better than $n + \lg n$.

Problem 0-10. Let's generalize the result from the last problem. Let $X : S \to \mathbf{N}$ and $Y : S \to \mathbf{N}$ be independent random variables and let $R = X + Y$ be their sum.

(a) Show that the probability generating function $G(x)$ for R is the product of those for X and Y. Is true in general if X and Y are not independent?

(b) Recall that $\mathrm{Ex}[R] = G'(1)$ where G' denotes the derivative with respect to x. Verify that this gives $\mathrm{Ex}[R] = \mathrm{Ex}[X] + \mathrm{Ex}[Y]$.

(c) Show that $G''(1) = \mathrm{Ex}[R^2] - \mathrm{Ex}[R]$.

(d) Deduce that $\mathrm{Var}(R) = G''(1) + G'(1) - (G'(1))^2$.

Problem 0-11. Let's use the last result of the last problem to compute the variance of "the time until the first success". If each trial has probability p of success, then random variable $R : S \to \mathbf{N}$, denoting the trial in which the first success is obtained, has distribution $\mathrm{Prob}[R = k] = (1-p)^{k-1}p$.

(a) Compute the probability generating function $G_R(x)$ and check that $G_R(1) = 1$.

(b) Compute $\mathrm{Var}(R)$ as outlined in the previous problem.

Problem 0-12. Recall Problem 33 on page 492 of [MR], which was problem **O9-8** on Problem Set 9. You were asked to compute the expected sum of k balls taken randomly *without* replacement from an urn of n balls numbered 1 through n. By linearity of expectation, this may be computed as the sum of the expected values of each of the balls taken from the urn. Each ball is equally likely to be any of the n, and so has expected value $\frac{1}{n}(1 + 2 + \cdots + n) = \frac{n+1}{2}$, so the expected sum is $k(n+1)/2$ (just as the case *with* replacement).

Solve this problem in a different way, using n indicator variables B_i, one for each of the n balls in the urn, to indicate the event that ball i is one of the k balls selected.

(a) Write an equation for the sum of the k balls using the indicator variables B_i.

(b) Compute $\mathrm{Prob}[B_i = 1]$.

(c) Now use parts **(a)** and **(b)** to compute the expected value of the sum. (Of course you should get the same answer!)

Problem 0-13. A cadet at the Naval Academy takes a multiple-choice test with 100 questions on which he gets 2 points for every correct answer and -1 point for every incorrect answer. He has mastered the material to only a small extent; he has a strategy for making educated guesses in such a way that his expected score is 50. Luckily, he has obtained a copy of a portion of the exam in advance, so he will not score less than -50. We don't know how the cadet is making his guesses, so we don't know anything about the distribution his score obeys except that the expected value is 50 and that his score is always in the range -50 to 200.

(a) Recall from lecture the converse of Markov's Inequality:

For random variable $R : S \to \mathbf{R}$, if R is always at most L, then

$$\text{Prob}[R > k] \geq \frac{\text{Ex}[R] - k}{L - k}.$$

Prove this using a proof by contradiction: assume the conclusion is false and reach a contradiction by showing that $\text{Ex}[R] < \text{Ex}[R]$.

(b) To repeat, we don't know what distribution the cadet's score obeys, but we can say something about it from the information we have.
What is a lower bound on the probability that the cadet scores more than 20?

(c) What is a lower bound on the probability that the cadet scores more than 50? Why is this? What distribution on the cadet's score achieves this probability?

(d) What is an *upper bound* on the probability that the cadet scores less than 0?

(e) What is an upper bound on the probability that the cadet scores less than 50?

(f) What is a *lower bound* on the probability that the cadet scores less than 100?

(g) Suppose we are told that the cadet guesses in such a way that the variance of the distribution on his score is 625. What is an upper bound on the probability that he scores at least 100?

(h) Suppose we are told that the cadet guesses in such a way that the sixth *central* moment of the distribution on his score is $1,000,000$. What is an upper bound on the probability that he scores at least 100?
You should do this by imitating the proof of Chebyshev's Theorem: Write

$$\text{Prob}[R - \text{Ex}[R] > 50] \leq \text{Prob}[(R - \text{Ex}[R])^6 > 50^6]$$

and use Markov's Inequality to get the upper bound for the latter.

Problem 0-14. Recall from recitation the baseball-card-collecting problem, where you buy cards one at a time and each is uniformly distributed from the entire set of n possible cards. We calculated the expected number of cards you will buy before collecting the entire set. To do this, we let random variable R_i denote the number of cards you buy to get a new card if you already have i different cards. The total number of cards you buy is therefore $R = \sum_{i=0}^{n-1} R_i$ and $\mathrm{Ex}[R] = \sum \mathrm{Ex}[R_i]$. R_i obeys the distribution from the last problem with $p = \frac{n-i}{n}$, so $E[R_i] = \frac{n}{n-i}$. So

$$\mathrm{Ex}[R] = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n\left(\frac{1}{n} + \cdots + 1\right) \sim n \ln n.$$

Calculate the variance of R using the calculation from the last problem and the fact that $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ for X, Y independent.

Problem 0-15. Suppose you are offered a betting game in which you keep flipping a fair coin until you get a head and if you flip the coin k times without getting a head, you win \$$2^k$. If the game costs \$1 to play, what is your expected return (net winnings)?
Would you be willing to play this game? (If so, please stop by NE43-370 any time and ask for Ravi.) Would you be willing to play this game if it cost \$1000? (If so, please stop by NE43-371 any time.)

Problem 0-16. Suppose you flip a fair coin 100 times. What is an upper bound on the probability that the number of heads is at least 70...

(a) ...according to Markov's Inequality?

(b) ...according to Chebyshev's Inequality?

(c) ...according to the upper bound for binomial distributions derived in recitation? (Get a strict upper bound by using the multiplicative factors of $\frac{1-\alpha}{1-2\alpha}$ to account for the case that there are strictly more than 70 heads, and $(\frac{n}{e})^{\frac{1}{12n}}$ to adjust Stirling's approximation.)
(This part requires a calculator, which won't be allowed on the exam.)

Problem 0-17. When designing VLSI circuits, a good rule of them is that the probability that the circuit will work is e^{-pA} for some constant p, where A is the area of the circuit. Give a justification for this rule under the assumption that every unit of area on the chip fails independently with probability p and that the circuit will fail if any unit of area fails.