Practice Quiz 2

- Solutions to this practice quiz will be available in the regular 6.042 drawer outside NE43-201 on Tuesday, April 19.
- This quiz contains 12 problems. You have 120 minutes to earn 125 points.
- **Initial the top of each page of your quiz. This is worth 5 points.**
- This quiz is closed book. You may use one $8\frac{1}{2}'' \times 11''$ handwritten crib sheet. No calculators or electronic playback devices are permitted.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

Name: ______________________________

Problem	Points	Grade
initials	5	
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	125	

Problem 0-1. [10 points] Suppose you are playing a game that results in either a win or a loss. If you play 4 times, what is the probability that you will win exactly three times? Assume that the outcomes of the games are mutually independent and that you win a game with probability 1/3.

Describe the sample space and show your calculations.

Problem 0-2. [10 points] Suppose you are shown two fair coins and a two-headed coin. They are then separated and hidden from you and you're allowed to pick one coin at random. Once you've picked the coin, it is flipped twice for you and you're allowed to view the results. Supposing that the coin turns up heads both times, what is the probability that it's the fair coin? Assume you can't cheat, that is you have probability $2/3$ of picking a fair coin, and that the tosses of a fair coin are independent.

Describe the sample space and show your calculations.

Problem 0-3. [10 points] Consider the following version of the Monty Haul game. The game is played normally, with one prize and three doors and Monty picks the box for the prize randomly. In this game however, Monty always opens the lowest numbered box not containing a prize. Analyze the following two strategies.

Strategy A: You always pick box 1. When Monty offers you the choice to switch you always accept it.

Strategy B: You always pick box 1. You switch only if Monty reveals box 3.

(a) What is the probability of winning for each strategy assuming Monty reveals box 3?

(b) What is the probability of winning for each strategy assuming Monty reveals box 2?

(c) What is the overall probability of winning for each strategy?

Problem 0-4. [10 points] Consider the following experiment. You roll a fair six sided die three times. Assume that the three rolls are mutually independent.

Let A_1 be the event that the first roll comes up 1.

Let A_2 be the event that the sum of the first two rolls is even.

Let A_3 be the event that the second roll is divisible by 3.

Let A_4 be the event that sum of all three rolls is at most 4.

Let A_5 be the event that the last roll comes up 1.

(a) What is the sample space?

(b) What are the probabilities of each atomic event?

(c) Are A_2 and A_4 independent?

(d) Find 3 events (from $A_1, \ldots, A_5$) that are mutually independent. (Prove your answer.)

(e) Is any event (again, one of $A_1, \ldots, A_5$) not independent of all the other events? (Prove your answer.)

Problem 0-5. [10 points] How many orderings exist for a standard card deck of 52 cards if all of the spades must appear in order but not necessarilly in a single block?

You may leave your answer in terms of factorials and powers.

Problem 0-6. [10 points] In how many ways can 12 people be seated at a square table, 3 on each side? Since the table is square, rotating their positions so that the people on the corners stay on the corners does not count as a new seating arrangement.

You may leave your answer in terms of factorials and powers.

Problem 0-7. [10 points] Let n and k be positive integers. What is the number of ways to distribute n distinct toys among k distinct children such that each child has at most 1 toy?

Problem 0-8. [10 points] Use a combinatorial argument to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

for any natural number n.

Hint: $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$.

Problem 0-9. [10 points] Use inclusion-exclusion to determine the number of 10-permutations of the multiset $\{4 \cdot a, 4 \cdot b, 4 \cdot c, \infty \cdot d\}$.

You may leave your answer in the form of a sum or difference of powers and binomial coefficients.

Problem 0-10. [10 points] Let a_n be the number of n-element multisets composed of the letters a, b, and c such that the number of a's is even, the number of b's is odd, and the number of c's is odd.

(a) Find the ordinary generating function for the sequence $\langle a_0, a_1, a_2, \ldots \rangle$.

(b) Find a relatively simple formula for a_n.

Problem 0-11. [10 points] Use generating functions to solve the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} & \text{for } n \geq 2. \end{cases}$$

For partial credit, just find a simple expression for an ordinary generating function for the sequence $\langle a_0, a_1, a_2, \ldots \rangle$.

Problem 0-12. [10 points] Use the pigeonhole principle to argue that the decimal expansion of a rational number m/n is repeating. For example, $26199/99900 = .26145145145145\ldots$ State explicitly what are the pigeons and what are the holes.

(Hint: use the grade-school method of long division.)