Problem Set 8 Solutions

Problem W8-7. **(a)** Let "W" denote the event that the Red Sox win. The sample space along with the associated probabilities are given in the figure. We see that

$$\text{Prob}[\text{Red Sox win}] = \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{2}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{4}\frac{3}{4} + \frac{1}{2}\frac{1}{1}\frac{3}{4}\frac{1}{2} + \frac{1}{2}\frac{1}{1}\frac{1}{4}\frac{2}{3} = \frac{7}{12}.$$

(b)

$$\text{Prob}[W \mid \overline{Y_2}] = \frac{\text{Prob}[W \wedge \overline{Y_2}]}{\text{Prob}[\overline{Y_2}]} = \frac{\frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}\frac{2}{3}\frac{1}{4}\frac{3}{4} + \frac{1}{2}\frac{1}{1}\frac{1}{4}\frac{2}{3}}{\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{1}\frac{1}{4}} = \frac{11}{18}.$$

(c) This is defined to be $1/m$—in this case, $1/2$.

Problem W8-8. **(a)** Let $s_1, \ldots, s_p$ be the names of the people in the room. Our sample space ranges over all birthdays and all orders in which we choose the people. Let us represent the sample space as all pairs $(\overline{b}, \pi)$ where $\overline{b}$ is a p-tuple, $\overline{b} = (b_1, \ldots, b_p)$ with $b_i \in \{1, \ldots, n\}$ (denoting the birthday of person s_i) and π is a permutation of $\{1, \ldots, p\}$ (so $s_{\pi(i)}$ is the ith person we choose). The p-tuples are all equally likely—each has probability $1/n^p$—and no matter how the birthdays occur, we choose an ordering π of the people uniformly—each π has probability $1/n!$—so the probability of any pair $(\overline{b}, \pi)$ is $1/(n!\, n^p)$.

Consider the pairs containing some fixed permutation π_0. How many p-tuples $\overline{b}$, when paired with π_0, result in an sample point where the first t names are written without anyone responding? The birthdays of persons $s_{\pi_0(1)}, \ldots, s_{\pi_0(t)}$ must all be different from each other and different from everyone else's. This can happen in $n(n-1)\cdots(n-t+1)(n-t)^{p-t} = \frac{n!}{(n-t)!}(n-t)^{p-t}$ ways. No matter which permutation π we fix, there will always be this many p-tuples that can be paired with π to result in a "successful" sample point. Thus the probability is

$$\left(n! \cdot \frac{n!}{(n-t)!}(n-t)^{p-t}\right) \frac{1}{n!\, n^p} = \frac{n!\,(n-t)^{p-t}}{(n-t)!\, n^p}.$$

You may conclude that the permutation π did not affect the probability, and so it did not matter that we chose people randomly: If we don't know anything about anyone's birthday—each person's birthday is assumed to be uniformly distributed throughout the year—then fixing an ordering of the people does not bias the probability one way or the other.

(b) Using Stirling's approximation, this probability is

$$\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (n-t)^{p-t}}{\sqrt{2\pi(n-t)}\left(\frac{n-t}{e}\right)^{n-t} n^p} = \sqrt{\tfrac{n}{n-t}}\cdot e^{-t}\cdot\frac{n^{n-p}}{(n-t)^{n-p}} = \sqrt{\tfrac{n}{n-t}}\; e^{-t}\;(1-t/n)^{p-n}.$$

If $t = o(n)$, then $n-t$ goes to n as n gets large, and so $\sqrt{\frac{n}{n-t}}$ goes to 1, and the above expression is thus $\sim e^{-t}(1-t/n)^{p-n}$ ("$\sim$" with respect to n).

(c) We simply have

$$e^{-t}\left(e^{-\frac{t}{n}-\Theta\left(\frac{t^2}{n^2}\right)}\right)^{p-n} = e^{-t+\frac{t}{n}(n-p)+\Theta\left(\frac{t^2(n-p)}{n^2}\right)} = e^{-\frac{tp}{n}+\Theta\left(\frac{t^2(n-p)}{n^2}\right)} = e^{-\frac{tp}{n}}\left(e^{\Theta\left(t^2\frac{1}{n}-t^2\frac{p}{n^2}\right)}\right).$$

If $t = o(\sqrt{n})$, then $t^2\frac{1}{n}$ goes to 0 as n gets large; similarly if $t = o(n/\sqrt{p})$, then $t^2\frac{p}{n^2}$ also goes to 0 as n gets large. Thus, under these two assumptions on t, the probability goes to $e^{-tp/n}$ as n gets large. Notice that in order to say that the probability is $\sim e^{-pt/n}$ (with respect to n), we need that both terms inside the Θ go to 0; it is not sufficient that both terms merely be small compared to tp/n. Also notice that in our analysis we used $1-x = e^{\ln(1-x)}$ and $\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots$ for $|x| < 1$ and the fact that $\frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots = O(x^2)$ for $|x| < 1$.

(d) From part **(a)**, we have the probability equal to

$$\frac{n(n-1)\cdots(n-t+1)}{n^t}\cdot\frac{(n-t)^{p-t}}{n^{p-t}} = \frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\frac{362}{365}\cdot\frac{361}{365}\cdot\left(\frac{360}{365}\right)^{95} = 0.2624.$$

From part **(c)**, the asymptotic estimate is $e^{-\frac{tp}{n}} = e^{-\frac{5\cdot 100}{365}} = 0.2541$, which is within 4% of the correct answer.

Problem W8-9. **(a)** Prob[Exactly k heads] $= \binom{n}{k}p^k(1-p)^{n-k}$.

(b)

$$\frac{\text{Prob[Exactly } k \text{ heads]}}{\text{Prob[Exactly } k-1 \text{ heads]}} = \frac{n-k+1}{k}\cdot\frac{p}{1-p},$$

which is 1 for $k = p(n+1)$. For $k < p(n+1)$ it is greater than 1, and for $k > p(n+1)$ it is less than 1. Thus $k = p(n+1)$ is the most likely number of heads.

(c) Prob[No heads] $= (1-1/n)^n$ which goes to e^{-1} as n gets large.

Prob[One head] $= \binom{n}{1}\left(\frac{1}{n}\right)^1\left(1-\frac{1}{n}\right)^{n-1}$, so the ratio is $\left(1-\frac{1}{n}\right)$.