Massachusetts Institute of Technology | Handout 31
6.042J/18.062J: Mathematics for Computer Science | April 8, 1994
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Problem Set 8

Reading: Maurer & Ralston: Sections 6.1–6.3.

There are only a few oral problems this week, due in tutorial on Monday, April 11. The written problems are due next week in recitation, April 15.

Problem O8-1. Consider the *Lets Make a Deal* game, when host Monty Hall chooses a door to contain the prize with some non-uniform distribution—he hides the prize behind door number i with some probability p_i, for $i = 1, 2, 3$ and $\sum_i p_i = 1$. Suppose you, the contestant, (not knowing the p_i's) follow the strategy of selecting a door uniformly at random—you choose door i with probability $1/3$—and then switching doors when offered. Determine the probability that you will win the prize by listing all events in the sample space along with the probabilities that they occur.

Problem O8-2. Suppose instead that Monty Hall chooses a door behind which to hide the prize according to the uniform distribution (all $p_i = 1/3$), but you the contestant select a door according to some non-uniform distribution—you select door i with some probability p_i, where $i = 1, 2, 3$ and $\sum_i p_i = 1$—and then switch when offered. Again determine the probability that you will win the prize by listing all events in the sample space along with the probabilities that they occur.

Problem O8-3. Consider the original *Lets Make a Deal* game, where there are 4 doors instead of 3. Monty and you choose doors according to the uniform distribution; Monty opens 2 doors (other than the one you chose) to reveal goats and then offers you the chance to switch to the remaining unopened door; you switch doors. Again determine the probability that you will win the prize by listing all events in the sample space and the probabilities that they occur. What is the probability that you will win if there are n doors?

Problem O8-4. Consider the *Lets Make a Deal* game yet one more time (please!), this time with host Malicious Monty Hall, who wants you to lose. In particular, if you initially choose a losing door, he may decide to end the game right there, opening your door to reveal the goat you have "won", never giving you the chance to switch doors. If Monty does give you the chance to switch in this game, should you? What is the probability of winning if you play the don't-switch strategy? What if you play the switch strategy?

Problem O8-5. Every evening at the Boston Symphony Orchestra (winter, spring, and summer) Bostonians check their hats (and scarves and gloves) at the coatroom on their way into the concert hall. Every evening, the hats are handed back randomly. Show that if the number of hats checked is n, then the probability that no one gets his own hat back is $\sim 1/e$ as n gets large. (Hint: Derangements.)

Problem O8-6. What is the probability of getting exactly 9 cards of the same suit in a bridge hand (which consists of 13 randomly chosen cards)? Define a "5-card major" to be a hand with either exactly 5 spades or exactly 5 hearts or both. What is the proabability of getting a 5-card major?

Problem W8-7. Joe, the bookie, is attempting to establish odds on an exhibition (!) baseball game. From many years of experience, he has learned that his prime consideration should be the selection of the starting pitcher. The Red Sox have only two pitchers, one of whom must start: R_1, their best pitcher; and R_2, a worse pitcher. The other team, the Yankees, has only three pitchers, one of whom must start: Y_1, their best pitcher; Y_2, a worse one; and Y_3, an even worse one. By carefully weighting information from various sources, Joe has decided to make the following assumptions: The Yankees will not start Y_1 if R_1 does not start. If R_1 does start, the Yankees will start Y_1 with probability 1/3. The Red Sox are equally likely to start R_1 or R_2, no matter what the Yankees do. Y_2 will pitch for the Yankees with probability 3/4 if Y_1 does not pitch, no matter what else occurs. The probability the Red Sox will win if R_1 pitches is $m/(m+1)$ where m is the number (subscript) of the Yankee pitcher. The probability the Yankees will win if R_2 pitches is $1/m$, where again m is the number of the Yankee pitcher.

(a) What is the probability that the Red Sox will win?

(b) If Y_2 does not start, what is the probability that the Red Sox will win?

(c) If R_2 and Y_2 start, what is the probability that the Yankees will win?

Problem W8-8. Assume everyone's birthday is randomly and uniformly chosen out of the n days in a year. Suppose we have a room of p people, and we repeatedly choose a person uniformly at random and write his or her birthday on a blackboard. If someone has the same birthday, he or she may shout out and we end the game. If not, we randomly choose another person and write his or her birthday on the board.

(a) Show that the probability that we will write t names on the board without matching anyone's birthday is

$$\frac{n!(n-t)^{p-t}}{(n-t)!n^p}$$

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(b) Use Stirling's approximation to show that this probability is

$$\sim e^{-t}(1-t/n)^{p-n}.$$

(c) Use the approximation $1 - t/n \approx e^{-\frac{t}{n}-\Theta(\frac{t^2}{n^2})}$ to show that the probability is approximately

$$e^{-\frac{tp}{n}+\Theta(\frac{t^2(n-p)}{n^2})}$$

and that if $t = o(p)$, then the probability is $\sim e^{-\frac{tp}{n}}$.

(d) Calculate the exact value of this probability for $n = 365$, $p = 100$, and $t = 5$. How does this compare to the asymptotic estimate with these values?

Problem W8-9. A biased coin has probability p of heads and probability $1 - p$ of tails. Suppose we flip this coin n times.

(a) What is the probability of getting exactly k heads?

(b) For what values of k is the probability of getting exactly k heads greater than the probability of getting exactly $k - 1$ heads? For what values of k is it less? What value of k is the most likely number of heads?

(c) Show that if $p = 1/n$ then the probability of getting no heads is $\sim 1/e$ (for large n). What is the ratio of the probability of no heads to the probability of exactly one head?