Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 26
March 29, 1994

Problem Set 7

Reading: Sections 2.1–2.3, 2.5 of Liu (handout 27).

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**"). The oral problems are due in tutorial on Monday, April 4. **The written problems are due at the beginning of class on Thursday, April 7.** Remember to write up each problem on a separate page or pages, with your name on each sheet.

Problem O7-1. Calculate the coefficients of each of the following series.

(a)

$$\frac{1}{(1+x)^m}$$

where m is an natural number.

(b)

$$\ln(1+3x)$$

(c)

$$x^5 \sin(x^2)$$

(d)

$$x^2 + \frac{1}{1-x}$$

(e)

$$(x+5)^4$$

(f)

$$\frac{1}{\cos(x)}$$

You only need to calculate the first 4 terms of this series.

Problem O7-2. Compute (ordinary) generating functions for each of the following sequences.

(a)

$$(1, 1, 1, 0, 0, 0, 0, 0, 0, \ldots)$$

(b)

$$(0, 0, 0, 1, 1, 1, 1, 1, \ldots)$$

(c)

$$(0, 0, 0, 1, 1, 2, 1, 1, 1, 1, 1 \ldots)$$

(d)

$$(0, 0, 0, 1, 2, 4, 5, 6, 7, 8, 9 \ldots)$$

(e)

$$(a_k) = \left(\frac{2}{k!}\right)$$

(f)

$$(a_k) = \left(\frac{2^k}{k!}\right)$$

(g)

$$(a_k) = \left(\frac{2^k}{(k-3)!}\right)$$

(h)

$$(1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$$

(i)

$$(0, 2, 0, 0, 2, 0, 0, 2/2!, 0, 0, 2/3!, 0, 0, 2/4!, \ldots$$

(j)

$$(0, 2, 0, 2, 0, 2/2!, 0, 2/3!, 0, 2/4!, \ldots)$$

(k)

$$(0, 2, 0, -2/3!, 0, 2/5!, 0, 2/7!, 0, 2/9!, \ldots)$$

(l)

$$(1, 4, 9, 16, 25, \ldots)$$

(m)

$$(2, 6, 12, 20, 30, 42, 56, \ldots)$$

(n)

$$(a_k) = \sum_{i=0}^{k} \frac{1}{i!}$$

(o)

$$(a_k) = 2^k + 3^k$$

Problem O7-3. Find partial fraction decompositions for the following generating functions.

(a)

$$\frac{x}{4x^2 - 1}$$

(b)

$$\frac{x^7 + 1}{4x^2 - 1}$$

(c)

$$\frac{x - 1/2}{4x^2 - 1}$$

(d)

$$\frac{x}{x^2 + x + 1}$$

(e)

$$\frac{(x-1)^2}{x^2 + x + 1}$$

(f)

$$\frac{x - 3}{x^3 + x^2 - x - 1}$$

(g)

$$\frac{1}{x^4 - 1}$$

(h)

$$\frac{x^3}{x^4 - 1}$$

Problem O7-4. Find (exactly) the coefficient of x^n in

$$\frac{x^4}{x^4 - 1} \quad .$$

Find a simple asymptotically tight ($\sim$) formula for the coefficient of x^n.

Repeat the above for the generating function

$$\frac{x^4 + x^2}{(x-1)^4} \quad .$$

Problem O7-5. Suppose we are buying party balloons which come in 8 different colors called (imaginatively) $C_1, \ldots, C_8$. Now, balloons of these colors are sold only in packets of size $2, 16, 2, 12, 12, 4, 4, 12$ respectively. What's more, the store you've gone to is down to its last packet of each color. I.E. if you want color C_4, you can only buy 0 or 12 ballons. Find a generating function for the number of ways to have an assortment of n colored balloons.

Problem O7-6. A "crooked die" is marked with faces $1, 2, 3, 4, 5, 7$. Let $f(k)$ be the number of ways in which the k distinguishable crooked dice can thrown so that their total is an even number. Find a simple formula for the ordinary generating function

$$\sum_{k \geq 0} f(k) x^k \quad .$$

Problem O7-7. Let $g(k)$ be the number of n-digit words formed from letters "a,b,c,d.e" such there are in total an even number of of letters "a" and "b." Find a simple formula for the ordinary generating function

$$\sum_{k \geq 0} g(k) x^k \quad .$$

Problem O7-8. Let $h(k)$ be the number of n-digit words formed from letters "a,b,c" such that no letter appears exactly three times Find a simple formula for the ordinary generating function

$$\sum_{k \geq 0} h(k) x^k \quad .$$

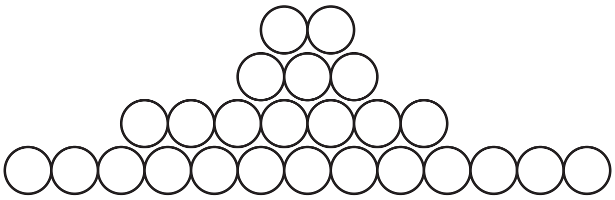

Figure 1: A block fountain with base of length 13.

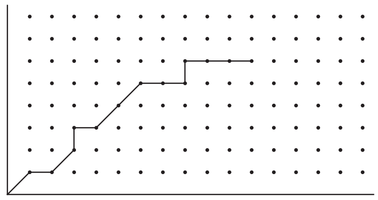

Figure 2: A lattice path from $(0,0)$ to $(11,6)$

Problem O7-9. A **block fountain** is an arrangement of rows of coins (or any other uniformly sized disks) such that each row is a contiguous block. Further, if a coin is in any row besides the bottom, then it rests on two coins in the row below it. See Figure 1. This problem counts the number of block fountains with base row of length k. Call this number $f(k)$.

(a) By breaking fountains up by removing their bottom rows, show that $f(k)$ satisfies

$$f(k) = 1 + \sum_{j=1}^{k} (k-j) f(j)$$

for $k \geq 1$, and $f(0) = 1$.

(b) Use the above recurrence to find a closed form for the generating function

$$F(x) = \sum_{i \geq 1} f(i) x^i \quad .$$

(c) Find a partial fractions decomposition of $F(x)$.

(d) Find a closed form formula for $f(k)$. Calculate the first 6 values. Do they look familiar? Derive a simple asymptotically tight ($\sim$) bound on the number of fountains with base k.

Problem O7-10. A **lattice path** is a path formed by straight lines in the plane such that each line starts and ends on a point in $\mathbf{N} \times \mathbf{N}$. This problem counts the number of lattice paths from the origin to another lattice point subject to the restriction that each line segment must come from a prescribed set. For example, Figure 2 is a lattice path with segments chosen from the set $\{(0,1),(1,0),(1,1)\}$.

(a) Let $f(m,n)$ be the number of lattice paths from $(0,0)$ to (m,n) using only steps from $\{(0,1),(1,0)\}$. Find a formula for

$$\sum_{m,n\geq 0} f(m,n)x^m y^n \quad .$$

Can you find a simple combinatorial interpretation for $f(m,n)$?

(b) Let $f(m,n)$ be the number of lattice paths from $(0,0)$ to (m,n) using only steps from $\{(0,1),(1,0),(1,1)\}$. I.E. this is the case from Figure 1. Find a formula for

$$\sum_{m,n\geq 0} f(m,n)x^m y^n \quad .$$

(c) Let $f(m,n)$ be the number of lattice paths from $(0,0)$ to (m,n) using only steps from $\{(0,3),(1,0),(4,5)\}$. Find a formula for

$$\sum_{m,n\geq 0} f(m,n)x^m y^n \quad .$$

(d) Do you see a pattern?

Problem O7-11. Define the backwards difference operator on functions from the natural numbers to the complex numbers by

$$\nabla f(n) = \begin{cases} f(n) - f(n-1) & \text{if } n > 0 \\ f(0) & \text{when } n = 0 \end{cases} \quad .$$

(a) Show if $F(x)$ is the (ordinary) generating function for $f(n)$, then $(1-x)F(x)$ is the generating function for $\nabla f(n)$.

(b) Show that if $F(x) = P(x)/(1-x)^{d+1}$ where $P(x)$ is a polynomial of degree less than or equal to d, then $F(x)$ is the generating function for some polynomial of degree d. (It turns out the converse is true as well.)

(c) Show that if $f(n)$ is a polynomial, then $\nabla^k f(n) = 0$ for sufficiently large k and n.

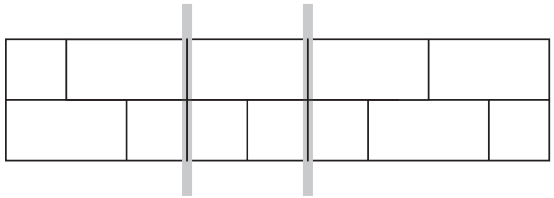

Figure 3: A tiling divided into 3 disjoint tilings of rectangles.

Problem W7-12. Suppose the generating functions $\sum_{i\geq 0} a_i x^i$ and $\sum b_i x^i$ can be written as $p(x)/q(x)$ and $r(x)/s(x)$ respectively for some polynomials p, q, r, s. It is an elementary (but nonetheless fairly deep) fact that $\sum_{i\geq 0} a_i b_i x^i$ can also be written as a ratio of polynomials. Thus we now a priori that if f_i is the ith Fibonacci number, then

$$\sum_{i\geq 0} f_{i+1}^2 x^i = 1 + 4x + 9x^2 + \cdots$$

can be written as a ratio of polynomials. This problem develops this generating function explicitly by examining certain tilings of rectangles. For the purposes of this problem, tiles cannot be rotated.

(a) Give a combinatorial argument that the number of ways to tile a $2 \times n$ rectangle with 1×2 and 2×1 tiles is F_{n+1}. Deduce that the number of ways to tile a $1 \times n$ rectangle by 1×1 and 1×2 tiles is also F_{n+1}. Further conclude that the number of ways to tile a $2 \times n$ rectangle by 1×1 and 1×2 tile is F_{n+1}^2.

(b) It is possible to divide some such tilings into disjoint parts by vertical lines. (Figure 3).

Argue briefly that the only tilings that cannot be divided into disjoint parts are those pictured in Figure 4.

(c) Find a closed form for the generating function $\sum_{i\geq 0} f_{i+1}^2$.

(d) From your formula for $\sum_{i\geq 0} f_{i+1}^2$, derive a simple asymptotically tight ($\sim$) bound on f_{i+1}^2. Does this agree with the asymptotic formula for f_{i+1}?

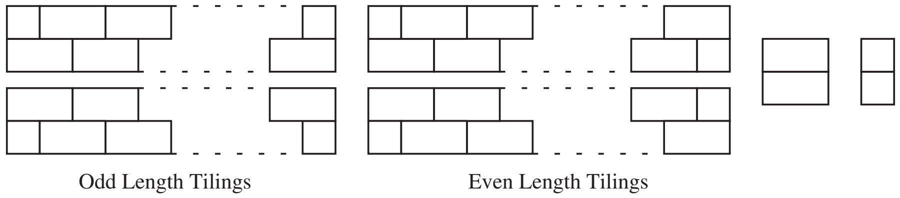

Figure 4: The ways to tile a rectangle so that it cannot be divided into tilings of disjoint rectangles. There are two special cases, one for length 1 tilings and one for length 2 tilings. There are 4 infinite families of tilings, two for tilings of even length (≥ 2) and o
and two for tilings of odd length (≥ 3).

Problem W7-13. A **partition** of a natural number n into k parts is a sequence of positive integers, $n_1 \le n_2 \le \cdots \le n_k$, such that $n = \sum_{i=1}^{k} n_i$. A part m is said to occur with **multiplicity** j if precisely j of the $n_1, n_2, \ldots, n_k$ are equal to m. For example $1, 2, 4, 4, 6$ is a partition of 17 into 5 parts; 4 has multiplicity 2 and all other parts have multiplicity 1.

(a) Let $p_1(n, k)$ be the number of partitions of n into k distinct parts–that is no part occurs with multiplicity greater than 1. Prove that

$$\sum_{n,k\ge 0} p(n,k) x^n y^k = \prod_{r\ge 1} (1 + yx^r) \quad .$$

(b) Prove that setting $y := 1$ in the generating function for $p_1(n, k)$ gives the (ordinary) generating function for the number, call it $p_1(n)$, of partitions of n with distinct parts. Namely,

$$\sum_{n\ge 0} p_1(n) x^n = \prod_{r\ge 1} (1 + x^r) \quad .$$

(c) Find the coefficients of $x^i y^j$ in the above generating function for $i, j = 0 \ldots 3$.

(d) Find an (ordinary) generating function for the number, $p_2(n)$, of partitions of n into odd parts. Use this to show that the number of such partitions is the same of the number of partitions of n into distinct parts.

(e) Let the expression $p(n)$ denote the number of partitions of n. Write down (and justify) an ordinary generating function for $p(n)$.

(f) Generalize the above formulas. Namely, suppose that M and R are two sets of positive integers. Let $p_{M,R}(n, k)$ be the number of partitions of n into k parts such that each part is in R and each part occurs with a multiplicity $m \in M$. Find a simple expression for

$$\sum_{n,k\ge 0} p_{M,R}(n,k) x^n y^k \quad .$$

Problem W7-14. This problem asks for a solution (in generating functions) to the question of how many ways can n distinguishable beads be strung together to form a collection of necklaces. In particular, these are real necklaces, so reversing the order of beads on a necklace doesn't make a new necklace. In real life this corresponds to flipping the necklace over. Furthermore, let's say we can't make a necklace from only one or two beads. Call this number f_n.

(a) Give a combinatorial argument for the fact that there are $(n-1)!/2$ different single necklaces made from $n \ge 3$ beads. What happens to this argument if $n \le 2$?

(b) Show that if $F(x)$ is an exponential generating function for the number of single necklaces on n beads, then $(F(x))^k$ is the exponential generating function for the number of ways to build a collection of k necklaces from a total of n beads.

(c) Find an exponential generating function in the variable x such that the coefficient of x^n is the number of ways to form k necklaces out of n beads.

(d) Again, suppose we want to build exactly k necklaces. Find an exponential generating function counting the number of ways to do this with n or fewer beads.