Massachusetts Institute of Technology Handout 25
6.042J/18.062J: Mathematics for Computer Science March 15, 1994
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Problem Set 6

Reading: Maurer & Ralston, Chapter 4.

The problems are divided into oral problems (prefixed "**O**") and written problems (prefixed "**W**"). The oral problems are due in tutorial on Monday, March 28. The written problems are due at the beginning of class on March 29.

Problem O6-1. Show that in any set of 12 integers, there are two whose difference is divisible by 11. [*Hint:* First show that any set of 3 integers has two whose difference is even.]

Problem O6-2. How many edges are possible in an undirected graph on n nodes? How many undirected graphs are possible on n nodes? (The nodes are labelled and therefore distinguishable, so $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ where $V = \{1, 2, 3\}$, $E_1 = \{(1, 2)\}$ and $E_2 = \{(1, 3)\}$ are different graphs.)

Problem O6-3. Let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$ and for any set X, let 2^X denote the set of all subsets of X. (Recall that the empty set and X itself are both subsets of X.) How many elements does $2^{[8]}$ contain? How many elements does $2^{[n]}$ contain? How many elements does $2^{2^{[8]}}$ contain?

Problem O6-4. In how many ways can we select a batting order of 9 ballplayers from a team of 24?

Problem O6-5. Evaluate $\binom{16}{4}$ and $\binom{17}{5}$. What is the ratio between $\binom{n+1}{k+1}$ and $\binom{n}{k}$?

Problem O6-6. Without using the formula for $\binom{n}{k}$, prove the identity

$$\binom{s-1}{0} + \binom{s}{1} + \cdots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n} = \binom{s+n}{n}.$$

where s and n are positive integers. (*Hint:* If X is an $(s+n)$-set[1] and $Y = \{y_1, y_2, \ldots, y_n\}$ is a specific n-subset of X, what is the number of n-subsets of X for which y_r is the first member of Y not in the subset?)

Problem O6-7. How many outcomes are there when 3 distinguishable dice are thrown? Show that there are only 56 possible outcomes when 3 indistinguishable dice are thrown. What is the number of possible outcomes when n indistinguishable dice are thrown?

[1]That is, a set containing $s + n$ elements

Problem O6-8. Calculate the coefficient of

(a) x^5 in $(1+x)^{11}$;

(b) a^2b^8 in $(a+b)^{10}$;

(c) a^6b^6 in $(a^2+b^3)^5$;

(d) x^3 in $(3+4x)^6$.

Problem O6-9. A computer operating system requires file names to be exactly four letters long; uppercase and lowercase letters are distinct.

(a) How many possible file names are there?

(b) The operating system is changed so that uppercase and lowercase are no longer distinct. Figure out the new number of file names in two ways: applying the division rule to **(a)** and directly.

(c) The operating system is changed so that file names may be *up to* four letters long. Repeat **(a)** and **(b)**. Can the division rule still be used?

Problem O6-10. How many ways can b indistinguishable balls be placed in u distinguishable urns if each urn must contain at least k balls? (Assume $k \le b/u$; otherwise it's impossible.)

Problem O6-11. In 6.042, 10 students are taking 6.002, 15 are taking 6.003 and 19 of the 44 students are taking neither. Also, 18 are taking 6.170, 3 of whom are also taking 6.002 and 6 of whom are also taking 6.003 (but none of whom is taking all 3.)[2]

(a) How many 6.042 students aren't taking any of these other three classes?

(b) How many are taking both 6.002 and 6.003?

(c) How many are taking 6.002 but not 6.003?

(d) How many are taking 6.003 but not 6.002?

(e) How many are taking 6.003 but neither of the other two?

(f) How many are taking 6.002, possibly 6.003, but not 6.170?

(g) How many are taking 6.002 and/or 6.170 but not 6.003?

Problem O6-12. Explain why an integer between 2 and 100 is prime if and only if either (a) it is a prime less than 10, or (b) no prime less than 10 divides it. Now use the Inclusion-Exclusion Principle to compute the number of primes less than 100. (You may know other methods that will answer this question faster, but if we replaced 100 by 1,000,000, this method would win.)

[2] These are the actual numbers!

Problem W6-13. How many ways can 4 balls be placed in 3 urns if

(a) both the balls and the urns are distinguishable;

(b) the balls are distinguishable but the urns are not;

(c) the urns are distinguishable but the balls are not;

(d) neither the balls nor the urns are distinguishable.

Problem W6-14. Derangements.

An inefficient secretary has n letters and n addressed envelopes for them. In how many ways can the secretary achieve the feat of putting every letter in the wrong envelope? This number is called the number of **derangements**, D_n.

(a) How many ways are there of achieving the feat with $n = 4$? Write down each of the different ways as a permutation of $\{1, 2, 3, 4\}$. (For example, 2341 represents putting letter number 2 in envelope number 1, letter 3 in envelope 2, letter 4 in envelope 3, and letter 1 in envelope 4.)

(b) The number of derangements D_n is the number of permutations of n elements that leave no element fixed. That is, it is the number of permutations π of $\{1, 2, \ldots, n\}$ such that $\pi(i) \neq i$ for all i. Let A_i denote the subset of permutations that leave i fixed. What is D_n in terms of n and the sets A_i?

(c) Let α_r be the cardinality of the intersection of the A_i taken r at a time, so

$$\alpha_r = \sum_{i_1 < i_2 < \cdots < i_r} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}|,$$

where the i_j are (distinct) integers between 1 and n. (Writing "sum over $i_1 < i_2 < \cdots < i_r$" is a slick way of saying "sum over all the different r-subsets $\{i_1, \ldots, i_r\}$".)
Express D_n in terms of n and the $\alpha_1, \ldots, \alpha_n$.

(d) Show

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

by computing α_r as the number of ways to choose the r integers times the number of ways of permuting the rest.

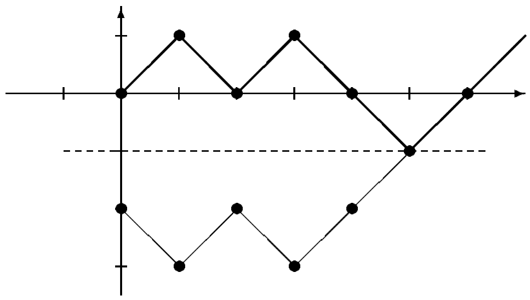
Problem W6-15. Catalan numbers.

In Problem Set 3, we saw a definition of well-formed strings of parentheses and brackets. We say a string of parentheses is well-formed if it is either ( ), (w), or $w\,v$, where w and v are well-formed. In this problem, you will count the number $C(n)$ of well-formed strings of parentheses containing n left-parentheses and n right-parentheses. The numbers $C(n)$ are called the **Catalan numbers**.

(a) What is the total number of strings (well-formed or not) containing n left-parentheses and n right-parentheses?

(b) There is a very clever argument for calculating $C(n)$ using something called **the reflection principle**. We will illustrate graphically the application of the reflection principle to this problem. Consider counting the number of ways of walking from the origin $(0,0)$ of the x,y-plane to the point $(2n,0)$. The walk must consist of exactly $2n$ steps, where each step goes right one unit and either up or down one unit. The figure below shows such a walk that happens not to go above $y=1$ for the first 7 steps. What is the number of ways of performing such a walk? Suppose now that you may never go below the x-axis. How is the number of such walks related to $C(n)$?

(c) Let us now count the number of walks that *do* go (strictly) below the x-axis. Argue that the number of such walks is equal to the number of walks that start instead at $(0,-2)$ and finish at $(2n,0)$ (but may cross the x-axis many times). Do this by finding a bijection between the two sets of walks (see figure) that justifies the term “reflection principle”.



(d) What is the number of walks that start at $(0,-2)$ and finish at $(2n,0)$ (but may cross the x-axis many times)?
Conclude that $C(n)$ is $\frac{1}{n+1}\binom{2n}{n}$.

(e) Let $B(n)$ be the number of binary trees on n nodes. (Having only one child as a left child is different from having only one child as a right child, so for example there are 5 binary trees on 3 nodes.) It turns out that $B(n)=C(n)$. Show that this is true either by (a) giving a bijection between the set of well-formed strings of parentheses (n left and n right) or (b) by showing that both $B(n)$ and $C(n)$ satisfy the recurrence

$$T(n) = \sum_{k=0}^{n-1} T(k)\cdot T(n-1-k)$$

where $n \geq 1$ and $T(0)=1$.