Problem Set 5

Reading: Maurer & Ralston: Chapter 5, Chapter 4 of CLR (handout 20).

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**"). Both the oral and written problems are due in tutorial on Monday, March 7. There is no lecture on March 8. Remember to write up each problem on a separate page or pages, with your name on each sheet.

Problem O5-1. Consider the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ aT(\frac{n}{2}) + n^2 & \text{for } n > 1 \end{cases}$$

Solve it for

(a) $a = 3.99$

(b) $a = 4$

(c) $a = 4.01$

Problem O5-2. Solve the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ 2T(\frac{n}{4}) + T(\frac{n}{2}) + n & \text{for } n > 1 \end{cases}$$

Problem O5-3. Get asymptotically tight upper and lower bounds for the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ T(\frac{n}{2} + 17) + T(\frac{n}{2} - 5) + \Theta(n) & \text{for } n > 1 \end{cases}$$

Problem O5-4. Solve the following recurrence (Hint: consider change of recurrence):

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ nT(\frac{n}{2}) & \text{for } n > 1 \end{cases}$$

Problem O5-5. Solve the following recurrence (Hint: simplify and substitute):

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ nT(\frac{n}{2}) + n & \text{for } n > 1 \end{cases}$$

Problem O5-6. Here is a recurrence that was once given in a 6.046 exam. Not only was no student able to do this problem, even the professor could not solve it. Try and give as tight a solution as you can:

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ T(n-1) + T(\frac{n}{2}) & \text{for } n > 1 \end{cases}$$

Problem O5-7. You have a computer which can do infinite precision arithmetic, i.e. it can multiply two irrational numbers exactly in one time step. Describe an $O(\log n)$ step algorithm on this computer to compute F_n, the nth Fibonacci number. (Hint: first figure out how to compute X^n in $O(\log n)$ steps for any X.)

Problem O5-8. An idealised *birth-death model* for bacteria. Each individual of a certain species of bacteria lives for two years. Each year each (alive) bacterium spawns off two copies of itself (making for a total of three).

(a) Let NB_n and OB_n denote the number of new-born and one-year old bacteria, respectively, at the end of year n. Set up recurrence relations for NB_{n+1} and OB_{n+1} in terms of NB_n and OB_n.

(b) Let P_n denote the number of bacteria alive at the end of year n. Set up a recurrence for P_n and solve it. Given that the species consists of one new-born bacterium at time 0, find the size of the population at the end of year 15.

Problem W5-9. **(a)** Solve the following recurrence (Hint: consider change of variables):

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ \sqrt{n}T(\sqrt{n}) + n & \text{for } n > 1 \end{cases}$$

(b) Consider the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{for } n = 1 \\ aT(\frac{n}{2}) + 1 & \text{for } n > 1 \end{cases}$$

Solve it for $a = .99$, $a = 1$ and $a = 1.01$.

Problem W5-10. You have n identical looking coins. All you know for sure is that more than half of them are made of gold. The rest of them are made of (possibly different) base metals. You have access to a machine which, given two coins, will tell you whether they are made of the same metal or whether they are made of different metals. But each query to the machine costs money. Devise an algorithm which determines all the gold coins in a linear (in n) number of queries. (Hint: make use of the recurrence relation $T(n) = T(\frac{n}{2}) + \frac{n}{2}$.)

Problem W5-11. The *graduate-student job problem* (made worse). Consider the following (depressing) scenario: Each year every professor spawns one new student who, after being a postdoc for one year goes on to become a professor the following year.

(a) Set up a recurrence relation for the number of professors and solve it. How fast does the number of professors grow? (Assume suitable initial conditions, but nontrivial—you may not assume there are no professors to start with)

(b) Now to compound the miseries of newly graduating students, in year n, $g(n)$ people switch from other careers to become professors without going to grad school. Set up the new recurrence and solve it for $g(n) = (\frac{3}{2})^n$ and $g(n) = 2^n$.

(c) If $g(n) = c^n$ then for what value of c will the influx of outsiders begin to make a significant difference to the growth rate of the number of professors (and wannabes)?