Problem Set 4

Reading: Maurer & Ralston: Sections 0.3-0.5, Chapters 2 and 3 of CLR (handout 14).

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**").[1] The oral problems are due in tutorial on Monday, February 28. The written problems are due at the beginning of class on Tuesday, March 1. Remember to write up each problem on a separate page or pages, with your name on each sheet.

Problem O4-1. The following is a list of functions. Identify for each function whether it is an injection, surjection, bijection (or none of the preceding.) If the function is not a bijection, indicate subsets of the domain and range so that the function is a bijection between these subsets.

(a) $f : \mathbf{R} \to \mathbf{R}$ such that
$$f : a \mapsto 3a$$

(b) $f : \mathbf{R} \to \mathbf{R}$ such that
$$f : a \mapsto \sin(a)$$

(c) $f : \mathbf{R}^+ \to \mathbf{R}$ such that
$$f : a \mapsto \log(a)$$

(d) $f : \mathbf{Z} \to \mathbf{Z}$ such that
$$f : a \mapsto \left\lfloor \frac{a}{3} \right\rfloor$$

Problem O4-2. Let X and Y be finite sets. Let $f : X \to Y$ be a function. Show that if f is injective then $|X| \leq |Y|$. Show that if f is a surjection then $|X| \geq |Y|$.

Prove that $|X| = |Y|$ iff there exists a bijection between X and Y.

Problem O4-3. Determine whether each of the following statements about the asymptotic growth of functions is true or false.

(a) $(x^2 + 3x + 1)^3 \sim x^6$

(b) $\frac{(\sqrt{x}+1)^3}{(x^2+1)} = o(1)$

(c) $e^{1/x} = \Theta(1)$

(d) $x^3(\log\log x)^2 = o(x^3 \log x)$

(e) $\sqrt{1 + \log x} = \Omega(\log\log x)$

[1] Thanks to H. Wilf's *Algorithms and Complexity* for providing a number of problems.

(f) $\sin(x) = \Omega(1)$

(g) $\sum_{j \le x} \frac{1}{j^2} = o(1)$

(h) $\sum_{j \le x} 1 \sim x$

Problem O4-4. The three sums listed below define three functions, h_1, h_2, h_3 on $x \in \mathbf{R}^+$.

$$h_1(x) = \sum_{1 \le j \le x} \left(\frac{1}{j} + \frac{3}{j^2} + \frac{4}{j^3} \right)$$

$$h_2(x) = \sum_{1 \le j \le x} (\log j + j)$$

$$h_3(x) = \sum_{1 \le j \le x} \left(\frac{1}{\sqrt{j}} \right)$$

For each sum, $h_i(x)$, find a closed form expression, $f_i(x)$, such that $h_i(x) = \Theta(f_i(x))$.

Problem O4-5. A function $f : \mathbf{R}^{\ge 0} \to \mathbf{R}$ is called **asymptotically nonnegative** if there exists a constant $x_0 > 0$ such that $f(x) \ge 0$ for all $x \ge x_0$.

There are six asymptotic symbols, namely $O, o, \sim, \Theta, \Omega, \omega$. Consider these symbols as relations on the set of functions $\{f : \mathbf{R}^{\ge 0} \to \mathbf{R} | f \text{ is asymptotically nonnegative}\}$. Prove that O is transitive. Namely, suppose $f, g, h : \mathbf{R}^{\ge 0} \to \mathbf{R}$ are asymptotically nonnegative. Show that if $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

Identify which of the reflexive, symmetric, antisymmetric, and transitive properties each relation satisfies. Which of the asymptotic relations are equivalence relations? You should be able to prove your statements.

Problem O4-6. A function $f : \mathbf{R}^{\ge 0} \to \mathbf{R}$ is called **asymptotically positive** if there exists a constant $x_0 > 0$ such that $f(x) > 0$ for all $x \ge x_0$.

Suppose $f, h : \mathbf{R}^{\ge 0} \to \mathbf{R}$ are asymptotically positive functions. We'll show essentially that there always exists a function g with rate of growth between that of f and h. Specifically, prove that if $f = o(h)$ then there exists a function g such that $f = o(g)$ and $g = o(h)$. Construct g explicitly in terms of f and h.

Problem O4-7. Prove that $\log(n!) \sim n \log n$. Prove that $n! = o(n^n)$.

Problem O4-8. A particular martial art has been studied by a set U of people and mastered by a set $M \subset U$ of people. Suppose there is a function $f : M \to M$ such that $m \in U$ was a student of $m' \in M$ if and only if $f(m') = m$. What can one deduce about the way this martial art is taught? If you were told that f was injective, what else could you conclude?

Problem O4-9. Let $f : A \to B$ and $g : B \to C$ be injective functions. Prove that $g \circ f : A \to C$ is also injective. If f and g are both surjective, is $g \circ f$ necessarily surjective? Prove your answer. What can you conclude about $g \circ f$ if f and g are bijections? If $g \circ f$ is a bijection, what can you conclude about g and f?

Problem O4-10. Prove that

$$1 + x \leq e^x \leq 1 + x + x^2$$

for $-1 \leq x \leq 1$.

Problem O4-11. Let $f : \mathbf{R}^{\geq 0} \to \mathbf{R}$ be asymptotically nonnegative. Prove or disprove the following statement:

$$f(x) = O\Big(g(x)\Big) \Longrightarrow f(x)^2 = O\Big(g(x)^2\Big)$$

As usual, if you disprove the statement, try to salvage as much as possible.

Problem W4-12. Rank the following fourteen functions in increasing order of their rates of growth. In particular, list them as $f_1, f_2, \ldots, f_{12}$ such that $f_i = O(f_{i+1})$. Indicate whether $f_i = o(f_{i+1})$ or $f_i = \Theta(f_{i+1})$.

$$2^{\sqrt{n}}, \quad e^{\ln(n^3)}, \quad e^{\ln^2(n^3)}, \quad \ln(3^{n^{\ln(n^3)}}), \quad n^{3.01}, \quad 2^{n^3}, \quad n^{1.6}$$

$$\ln(n^3+1), \quad \sqrt{n!}, \quad n^{3\ln n}, \quad n^3 \ln n, \quad (\ln\ln n)^3, \quad n^{.5}2^n, \quad (n+4)^{12}$$

Briefly justify your ranking.

Problem W4-13.

(a) Find a function $f : \mathbf{R}^{\geq 0} \to \mathbf{R}$ such that $f(x) = O(x^{1+\varepsilon})$ for every constant $\varepsilon > 0$, but for which $f(x) \neq O(x)$. Prove that your function actually satisfies these asymptotic properties.

(b) Let $a, b \in \mathbf{R}^+$ be constants. Prove that if $a > 1$, then $x^b = o(a^x)$.

(c) Let a be a constant greater than 0. Prove that $a^x = o((a+\varepsilon)^x)$ for any constant $\varepsilon > 0$.

(d) Let $f : \mathbf{R}^{\geq 0} \to \mathbf{R}$ be an asymptotically positive function. Fix a constant $\varepsilon > 0$. Suppose further that $f(x) = O((2+\varepsilon)^x)$ It is not necessarily true that $f(x) = o((2+\varepsilon)^x)$.

The statements "$f(x) = O((2+\varepsilon)^x)$ for all constants $\varepsilon > 0$" and "$f(x) = o((2+\varepsilon)^x)$ for all constants $\varepsilon > 0$" are, however, equivalent. Prove this.