Problem Set 4

Reading: Maurer & Ralston: Sections 0.3-0.5, Wilf(handout)?,

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**").

The oral problems are due in tutorial on Monday, February 28. The written problems are due at the beginning of class on Tuesday, March 1. Remember to write up each problem on a separate page or pages, with your name on each sheet.

Problem O1-1.

Problem W1-2. In this problem you are asked to prove a version of Stirling's approximation to the factorial function. The basic approach will be to bound the natural log function, written log, by two function, f and g, that have natural geometric interpretations in terms of $\log n!$ and can be easily integrated. The result will be a bound on $\log(n!)$, which will (by exponentiation) bound $n!$.

(a) Let f and g be functions from an interval $[a, b)$ on the real line to $\mathbf{R}$. Prove that if $f'(x) \geq g'(x)$ for all $a \leq x < b$ and if $f(a) = g(a)$ then $f(x) \geq g(x)$ for all $a \leq x < b$.

(b) Prove the dual proposition. Namely, let f and g be functions from an interval $(a, b]$ on the real line to $\mathbf{R}$. Prove that if $f'(x) \leq g'(x)$ for all $x \in (a, b]$ and if $f(b) = g(b)$ then $f(x) \geq g(x)$ for all $x \in (a, b]$.

(c) A function $f : (a, b) \to \mathbf{R}$ is said to be **c**onvex if for any points $a < x < y < b$ the function $l : (x, y) \to \mathbf{R}$ defined by

$$l(z) = \frac{z-y}{x-y} f(x) + \frac{x-z}{x-y} f(y)$$

satisfies $l(z) \leq f(z)$ for all $z \in (x, y)$.

Assume that f is (infinitely differentiable?) (taylorseries converges?) on (a, b). Prove that f is convex on (a, b) iff f'', the second derivative of f, is non-positive on (a, b).

(d) Define a function $f : [0, \infty) \to \mathbf{R}$ as follows. For all $m = 1, 2, 3, \ldots$, if $m \leq x \leq m+1$ then let

$$f(x) = (m+1-x)\log m + (x-m)\log(m+1) \quad .$$

Similarly, $g : [0, \infty) \to \mathbf{R}$ is defined in pieces. For all $m = 1, 2, 3, \ldots$, if $m - 1/2 \leq x < m + 1/2$ then let

$$g(x) = \frac{x}{m} - 1 + \log m \quad .$$

Graph f and g.

(e) Prove that for $x \in [0, \infty)$, $f(x) \leq \log x \leq g(x)$.

(f) Prove that

$$\int_1^n g(x) = log(n!) + \frac{1}{8} - \frac{1}{2}\log n - A$$

for some positve number A which might depend on n.

(g) Prove that

$$\int_1^n f(x) = log(n!)\frac{1}{2}\log n \quad .$$

(h) Prove that $\int \log u \, du = u \log u + C$.

(i) Show that

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2}) \log n + n < 1 \quad .$$

(j) Conclude that

$$e^{7/8} \left(\frac{n}{e}\right)^n \sqrt{n} < n! < e \left(\frac{n}{e}\right)^n$$

for $n = 2, 3, 4, 5, \ldots$. Compare this to Stirling's formula that

$$\sqrt{2\pi} \left(\frac{n}{e}\right)^n \sqrt{n} \sim n! \quad .$$

(k) Use Stirling's formula to obtain an asymptotic approximation to $\binom{n}{k}$ as n and k both approach infinity. Use your approximation to obtain bounds on $\binom{n}{k}$ for all n and k.

Problem W1-3. In this problem you are asked to examine some properties of power series.

(a) Let I_X and I_Y be the identity functions on the sets X and Y respectively. Let $f : X \to Y$ and $g : Y \to X$ be functions. Show that $f \circ g = I_X$ does not imply that $g \circ f = I_Y$.

(b) Prove that if $f \circ g = I_X$ and there exists h such that $g \circ h = I_Y$ then $g \circ f = I_Y$ and $f = h$.

(c) The **formal power series** in x with coefficients in $\mathbf{R}$ are all series $\sum_{i \geq 0} a_i x^i$ with $a_i \in \mathbf{R}$. I.E. we don't worry about convergence. This set is denoted $\mathbf{R}[[x]]$. Because formal power series do not need to converge, they are not functions on $\mathbf{R}$. However, they can be treated as functions. Let $x\mathbf{R}[[x]]$ denotet the set

$$\left\{ sum_{i \geq 0} a_i x^i \in \mathbf{R}[[x]] : a_0 = 0 \quad . \right\}$$

The notation arises from the fact that this set looks like what you should get from multiplying every formal power series by x.

A formal power series $f \in \mathbf{R}[[x]]$ is a function from $x\mathbf{R}[[x]]$ to $\mathbf{R}[[x]]$, as follows: Let $f = \sum_{i \geq 0} a_i x^i$. Let $g \in x\mathbf{R}[[x]]$ be $\sum_{i \geq 1} b_i x^i$. Then $f(g) = \sum i \geq 0 c_i x^i$ is defined by letting c_i be the coefficient of x^i in

$$a_0 + a_1 \left(\sum_{j=1}^{i} b_i x^i \right) + a_2 \left(\sum_{j=1}^{i} b_i x^i \right)^2 + \cdots + a_i \left(\sum_{j=1}^{i} b_i x^i \right)^i \quad .$$

Show that a formal power series $f \in x\mathbf{R}[[x]]$ is a function from $x\mathbf{R}[[x]]$ to $x\mathbf{R}[[x]]$. Is some formal power series in $x\mathbf{R}[[x]]$ the an identity function, $I_{x\mathbf{R}[[x]]}$, and if so which formal power series is it?

(d) Given $g, h \in x\mathbf{R}[[x]]$, show that $g \circ h = g(h)$.

(e) Given $g = \sum_{i\geq 1} a_i x^i \in x\mathbf{R}[[x]]$, show that there exists $h \in x\mathbf{R}[[x]]$ such that $g \circ h = x$ iff the $a_1 \neq 0$. Conclude that $h \circ g = x$.

(f) Formal power series can be multiplied by extending the distributive law as follows:

$$\left(\sum_{i\geq 0} b_i x^i\right) \star \left(\sum_{j\geq 0} c_j x^j\right) = \sum_{k\geq 0}\left(\sum_{i=0}^{k} b_i c_{k-i}\right) x^k \quad .$$

Suppose h and g are formal power series which, when treated as functions on $\mathbf{R}$ both converge absolutely in an interval (a_0, a_1). It turns out that $h \star g$ treated as a function also converges on (a_0, a_1), and $g(a)h(a) = g \star h(a)$ for all $a \in (a_0, a_1)$. Use this fact to prove that

$$\left(\sum_{i\geq 0} x^i\right) \star \left(\sum_{i\geq 0} (-1)^i x^i\right) = 1 \quad .$$

Verify the above relation directly from the definition of $\star$.