Massachusetts Institute of Technology | Handout 8
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Professors Charles E. Leiserson and Tom Leighton

Weak induction, strong induction, and the well-ordering principle

Last week in class, we stated the following theorem.

Theorem 1 *Weak induction, strong induction, and the well-ordering principle are equivalent.*

Some confusion arose in the proof of the theorem, which we hope to clear up with this handout.

The idea of the proof was to show the following:

$$\begin{aligned} \text{strong induction} &\Rightarrow \text{weak induction} \\ \text{weak induction} &\Rightarrow \text{well-ordering principle} \\ \text{well-ordering principle} &\Rightarrow \text{strong induction} \end{aligned}$$

The meaning of "$\Rightarrow$" here is the usual meaning of logical implication. For example, the first statement says that if strong induction is accepted as an axiom, then weak induction can be proved as a consequent theorem. By the transitivity of implication (syllogism), the cycle of implications proves that any one statement implies any other statement, and thus, that all the statements are equivalent. (This strategy is often useful for showing that several statements are equivalent.)

The second and third cases were proved in class in some detail. Some confusion arose, however, when Professor Leiserson glibly stated that the first case (strong induction $\Rightarrow$ weak induction) was "trivial." His justification was that if a theorem can be proved using weak induction, then it certainly can be proved using strong induction. A discussion ensued in which it was argued that this justification was for the converse (weak induction $\Rightarrow$ strong induction).

Professor Leiserson's justification is indeed correct, and what follows is a proof that strong induction implies weak induction. Indeed, the proof is more a lesson in logical reasoning than anything having to do with induction or well ordering, in contrast with the proofs of the other two cases. The proof took the staff of 6.042J/18.062J many hours to produce and verify, and now, we are all able to certify that it is indeed trivial. `:-)`

Let us begin by stating clearly what we wish to show using logical notation to resolve ambiguities. In this case, we wish to show that strong induction implies weak induction. Thus, for any predicate P defined on the natural numbers $\mathbf{N}$, we must show that

$$\begin{aligned} &\left[\Big(P(0) \wedge \big(\forall n\, P(0) \wedge P(1) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)\big)\Big) \Rightarrow \forall n\, P(n)\right] \\ &\quad \Rightarrow \left[\Big(P(0) \wedge \big(\forall n\, P(n) \Rightarrow P(n+1)\big)\Big) \Rightarrow \forall n\, P(n)\right] . \end{aligned}$$

In this expression, the domain of the "$\forall$" is the natural numbers $\mathbf{N}$.

To help us reason about this rather imposing logical formula, let us define

$$A = \Big[\forall n\; P(0) \wedge P(1) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)\Big]$$

and

$$B = \Big[\forall n\; P(n) \Rightarrow P(n+1)\Big] \ .$$

Thus, we wish to prove the validity of the formula

$$\Big(P(0) \wedge A \Rightarrow \forall n\; P(n)\Big) \Rightarrow \Big(P(0) \wedge B \Rightarrow \forall n\; P(n)\Big) \ .$$

Observe now that $B \Rightarrow A$, since A has a stronger hypothesis than B: if $P(n+1)$ follows from merely $P(n)$, then it follows from the stronger (more restrictive) hypothesis $P(0) \wedge P(1) \wedge \cdots \wedge P(n)$. The rule of propositional logic that we are using is essentially $(X \Rightarrow Z) \Rightarrow (X \wedge Y \Rightarrow Z)$, which you can verify using the truth-table method.

Since $B \Rightarrow A$, we have $P(0) \wedge B \Rightarrow P(0) \wedge A$. We also have $P(0) \wedge A \Rightarrow \forall n\; P(n)$, since we are assuming strong induction. Using transitivity of implication, we conclude that $P(0) \wedge B \Rightarrow \forall n\; P(n)$, which is the statement of weak induction. Thus, strong induction (together with the rules of logic) implies weak induction.