Problem Set 2

Reading: Cormen, Leiserson & Rivest: Chapter 5 (Handout 9). Maurer & Ralston: Section 0.1–3, Chapter 2, Sections 3.1–3.

The Athena course locker containing course handouts has been set up. Type `attach 6.042` at the Athena prompt and then `cd /mit/6.042` to get into the course directory. Once there, you can type `lpr -P⟨printer-name⟩ ⟨file-name⟩` to print the file of your choice. For example, to print this handout you would type `lpr -P⟨printer-name⟩ H07-Pset2.ps`.

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**"). The oral problems are due in tutorial on Monday, February 14. The written problems are due at the beginning of class on Tuesday, February 15. Remember to write up each problem on a separate page or pages, with your name on each sheet. Read Handout 1 for the policy on problem sets.

Problem O2-1. Let $G = (V, E)$ be an undirected graph, and let $r : V \to \mathbf{R}$ be a function associating a real number with each vertex. Suppose r satisfies the property that for each $v \in V$, the value $r(v)$ is the average of the values of v's neighbors. Prove that if the graph is connected, then $r(u) = r(v)$ for all $u, v \in V$.

Problem O2-2. Let $G = (V, E)$ be an n-vertex dag (directed acyclic graph). Show that each vertex $v \in V$ can be assigned a distinct number $r(v)$ from the set $\{1, 2, \ldots, n\}$ such that if $(u, v) \in E$, then $r(u) < r(v)$.

Problem O2-3. Prove that in any set of 101 distinct integers chosen from $\{1, 2, \ldots, 200\}$, there exist two such that one is an integer multiple of the other. Produce a set of 100 distinct integers in $\{1, 2, \ldots, 200\}$ such that no number in this set divides another.

Problem O2-4. Prove that the number of leaves in any rooted binary tree is one more than the number of nodes with degree 2. (Read the footnote concerning the degree of a node in a rooted tree on page 94 of Handout 9.)

Problem O2-5. Prove that edges of any connected, undirected graph with an even number of edges can be oriented so that every vertex has even out-degree.

Problem O2-6. Verify or falsify the following proof that a binary relation R that is symmetric and transitive is also reflexive, and hence, an equivalence relation. We have $aRb \Rightarrow bRa$ by symmetry, and thus, by transitivity, it follows that aRa.

Problem O2-7. Let S be a finite set, and let R be an equivalence relation on $S \times S$. Show that if R is antisymmetric, then the equivalence classes of S with respect to R are singletons.

Problem O2-8. Prove that any graph on 6 vertices contains 3 vertices with no edges between them or 3 vertices with all edges between them.

Problem O2-9. Prove that each vertex of an undirected graph can be colored red or grey such that every vertex is connected to at least as many neighbors of the opposite color as it is to neighbors that are the same color as itself.

Problem W2-10. Each vertex of an undirected graph $G = (V, E)$ contains a toggle switch and a lightbulb. If you push the toggle switch in a vertex, the bulb in the vertex goes off if it was on, and on if it was off. Moreover, all the bulbs in neighboring vertices are toggled as well.

Initially, all the bulbs are off. Notice that pushing the same switch twice doesn't affect any of the bulbs. Hence, we can talk of a "setting" of the switches, which corresponds to the subset of vertices whose switches have been pushed once (or an odd number of times). The aim of this problem is to prove that for any undirected graph G, there exists a **good** setting of the switches: a setting such that all the bulbs are turned on simultaneously.

(a) Suppose that G has an even number $|V|$ of vertices. Prove that if every graph on $|V| - 1$ vertices has a good setting, then a good setting for G can be obtained by combining (in some suitable way, which you have to come up with) the settings corresponding to the subgraphs induced (see Handout 9, page 88) by the $|V|$ subsets of V that have $|V| - 1$ vertices.

(b) Suppose that G has an odd number of vertices. Prove that there exists a vertex with even degree.

(c) Suppose that G has an odd number of vertices, let v be an even-degree vertex, and let S be the set of vertices that includes v and its neighbors. Prove that if every graph on $|V| - 1$ vertices has a good setting, then a good setting for G can be obtained by combining the settings corresponding to the subgraphs induced by some $|V| - |S|$ subsets of V that have $|V| - 1$ vertices.

(d) Conclude that a good setting exists for any undirected graph.

Problem W2-11. A **hamiltonian path** in an undirected graph is a simple path that visits all the vertices. An undirected graph is said to be **hamiltonian** if there exists a simple cycle that contains all the vertices. Let A be the adjacency matrix of an undirected graph G, where we assume $A[v, v] = 1$ for all $v \in V$. The **cube** of G is the graph $G^3 = (V, E^3)$ whose adjacency matrix is A^3, where we perform matrix multiplication using the boolean operations $\vee$ and $\wedge$ instead of the usual $+$ and $\times$. The aim of this problem is to show that the cube of any connected graph is hamiltonian.

(a) Prove that any connected graph contains a free tree that connects all the vertices as a subgraph.

(b) Argue briefly that $(u, v) \in E^3$, or equivalently, that $A^3[u, v] = 1$, if and only if the distance between u and v in G is at most 3.

(c) Prove by induction that the cube of any rooted tree contains a hamiltonian path that begins at the root and ends at one of its children.

(d) Prove that the cube of any free tree is Hamiltonian, and conclude that the cube of any connected graph is Hamiltonian.