Problem Set 1

Reading: Maurer & Ralston: Section 0.7, Chapter 2, Sections 7.1–3, and Sections 7.5-7.6.

The problems are divided into oral problems (prefix "**O**") and written problems (prefix "**W**").

The oral problems are due in tutorial on Monday, February 7. Some of the problems ask you to "verify or falsify" a proof. For these problems, you must either justify the correctness of each step of the proof, or else locate and explain the error. It is not sufficient to merely show that an incorrect proof is indeed wrong. You must explain where the bug is.

The written problems are due at the beginning of class on Tuesday, February 8. Remember to write up each problem on a separate page or pages, with your name on each sheet. Read Handout 1 for the policy on problem sets.

Problem O1-1. Verify or falsify the following proof that if a and b are two equal real numbers, then $a = 0$.

Proof.

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a+b &= b \\ a &= 0 \end{aligned}$$

$\square$

Problem O1-2. Verify or falsify the following proof that there exist two irrational numbers a and b such that a^b is rational.

Proof. Recall that $\sqrt{2}$ is irrational. Suppose that $\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational. Then, let $a = \sqrt{2}$, let $b = \sqrt{2}$, and the proof is finished. Suppose, however, that $\left(\sqrt{2}\right)^{\sqrt{2}}$ is irrational. Then, let $a = \left(\sqrt{2}\right)^{\sqrt{2}}$ and let $b = \sqrt{2}$. Thus, we have $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$, which is rational, and we're done. $\square$

Problem O1-3. Verify or falsify the following proof that for all real $x \neq 0$, and all natural numbers n, we have $x^n = 1$.

Proof. We use induction on n. For the base step, observe that if $x \neq 0$, we have $x^0 = 1$. We now assume that for all $k \leq n$, we have $x^k = 1$. We shall show that $x^{n+1} = 1$. By the induction hypothesis, we have $x^n = x^{n-1} = 1$. Hence, it follows that

$$\begin{aligned} x^{n+1} &= \frac{x^n x^n}{x^{n-1}} \\ &= \frac{1 \times 1}{1} \\ &= 1 . \end{aligned}$$

□

Problem O1-4. Verify or falsify the following proof that $\sum_{i=1}^{n} i$ is less than or equal to some positive constant times n.

Proof. The proof is by induction on n.

If $n = 1$, then the summation evaluates to 1, which can be bounded above by c times 1, where c is any constant greater than or equal to 1.

Assume that the statement is true for $n = k$, that is, that

$$\sum_{i=1}^{k} i \leq ck$$

for some constant $c \geq 1$. Adding $k + 1$ to both sides, we obtain

$$\begin{aligned} \sum_{i=1}^{k+1} i &\leq ck + (k+1) \\ &\leq (c+1)(k+1) . \end{aligned}$$

Since c is a constant greater than or equal to 1, so is $c + 1$. For $n = k + 1$, therefore, the sum $\sum_{i=1}^{n} i$ is also bounded above by a constant times n. Consequently, by induction this property holds for every natural number $n \geq 1$. □

Problem O1-5. Verify or falsify the following proof that there exists an obtuse angle with measure equal to that of a right angle.

Proof. Construct a rectangle $abcd$ as shown in Figure 1. Construct $\overline{ae}$ such that $\overline{ae}$ is the same length as $\overline{ac}$ and such that $\angle eac$ has small measure. Draw $\overline{ed}$. Now, construct perpendicular bisectors to $\overline{ed}$ and to $\overline{cd}$, and let h be their point of intersection. Construct $\overline{ah}$, $\overline{bh}$, $\overline{eh}$, and $\overline{dh}$.

Since the line $\overline{fh}$ is a perpendicular bisector of $\overline{ed}$, the triangle $\triangle ehd$ must be isoceles, and hence the lengths of $\overline{eh}$ and $\overline{hd}$ are equal. Similarly, since $abcd$ is a rectangle, $\overline{gh}$ is a perpendicular bisector of $\overline{ab}$, so $\triangle ahb$ is isoceles. Therefore, $\overline{ah}$ and $\overline{hb}$ are equal in length, and $\angle bah$ is equal in measure to $\angle hba$.

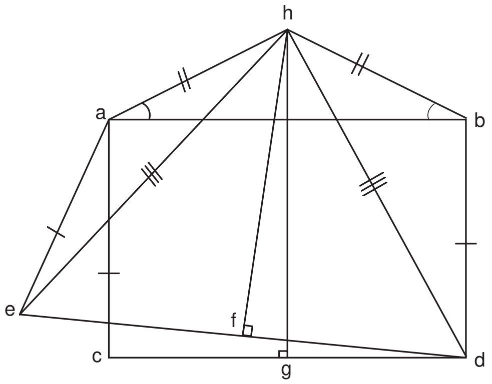

Figure 1: A sketch of the geometric construction in Problem O1-5.

Since the corresponding sides of $\triangle eah$ and $\triangle dbh$ have the same lengths, it follows that the two triangles are congruent. Consequently, $\angle eah$ and $\angle dbh$ have equal measure. Subtracting the (equal) measures of $\angle bah$ and $\angle abh$, respectively, we find that the measure of $\angle eab$ equals the measure of $\angle dba$. Thus, an obtuse angle has the same measure as a right angle. $\square$

Problem O1-6. Consider the three-dimensional integer lattice of all points in $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. Show that for any nine points in this lattice, there exist two points of the nine such that the midpoint of the straight line connecting them is also a lattice point.

Problem O1-7. Prove that for any natural number n,

$$\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21}$$

is an integer. (*Hint:* Use the polynomial expansion of $(n+1)^7$.)

Problem O1-8. You are given an unlimited supply of 4-cent stamps and 7-cent stamps. Obviously, with these two denominations, you cannot make certain amounts of postage, such as 10 cents. Prove, nevertheless, that for some $n_0 > 0$, every amount n of postage such that $n \geq n_0$ can be made up from 4- and 7-cent stamps. What is the minimum value for n_0 such that this theorem holds?

Problem O1-9. For any integer $n \geq 3$, consider a simple, not necessarily convex, polygon of n vertices drawn in the plane. (A polygon is **simple** if it does not cross itself.) An ***n*-gon** is a polygon with n vertices. A **triangulation** of an n-gon P is a partition of the interior of P into triangles such that the vertices of any such triangle are vertices of P.

(a) Use induction to show that any triangulation of an n-gon is comprised of exactly $n-2$ triangles.

(b) Using the fact the sum of the interior angles of a triangle is π, give a formula for the sum of the interior angles of an n-gon. Prove that your formula is correct.

Problem W1-10. Recall that the binomial coefficients $\binom{n}{k}$ ("read n choose k") are defined on the natural numbers by

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}\ .$$

The binomial coefficients are precisely the numbers appearing in Pascal's triangle, which is shown in Figure 2. In each row of Pascal's triangle, the first and last entries are 1. Any other entry is the sum of the two entries above it.

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&&\\
&&&&&&1&&1&&&&&&\\
&&&&&1&&2&&1&&&&&\\
&&&&1&&3&&3&&1&&&&\\
&&&1&&4&&6&&4&&1&&&\\
&&1&&5&&10&&10&&5&&1&&\\
&1&&6&&15&&20&&15&&6&&1&\\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

Figure 2: The first eight rows of Pascal's triangle.

(a) Justify the reading of the notation $\binom{n}{k}$ by using induction to prove that $\binom{n}{k}$ is the number of ways to choose k objects from a set of n distinct objects.

(b) Prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $n \geq 1$ and $0 < k < n$. Argue that the kth entry in row n of Pascal's triangle is $\binom{n}{k}$, if rows and entries are labeled suitably.

(c) Prove that each row of Pascal's triangle increases until the middle and then decreases.

(d) Give an inductive proof that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\ .$$

Use this fact to show that the sum of the entries in the nth row of Pascal's triangle is 2^n. Similarly, prove that the sum of the even-numbered entries minus the sum of the odd-numbered entries is 0 in any but the top row of Pascal's triangle.

(e) Consider taking the sum of all numbers in the "off-diagonals" of Pascal's triangle as shown in Figure 3. What sequence do they form? State your answer formally, and prove it.

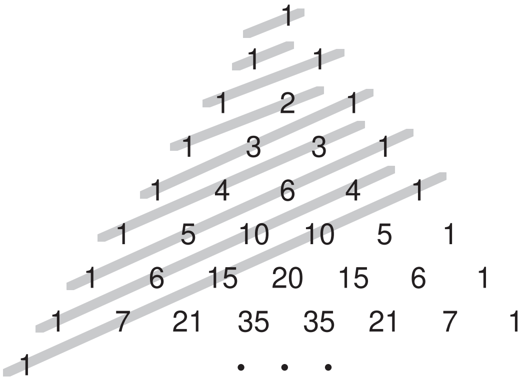

Figure 3: The "off-diagonals" of Pascal's triangle.

Problem W1-11. Consider a simple polygon drawn with vertices on the integer $\mathbf{Z} \times \mathbf{Z}$ lattices, as is shown in Figure 4. Let I be the total number of lattice points in the interior of the polygon, and let B be the total number of lattice points on the boundary. Prove that the area A of the polygon is $A = I + B/2 - 1$. (*Hint:* The proof will take you several steps.)

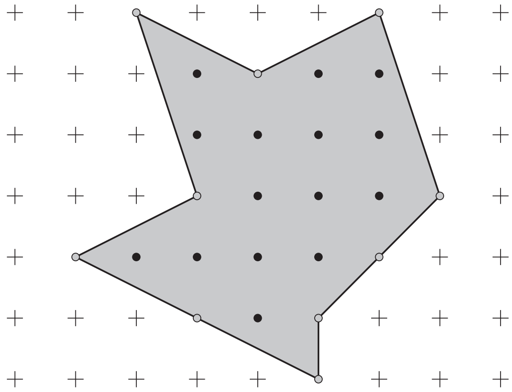

Figure 4: A simple polygon in the lattice $\mathbf{Z} \times \mathbf{Z}$, with 15 lattice points in its interior, 10 lattice points on its boundary and area 19.