Review Problems

Problem 1. (Negative Binomial Distribution) Let p be the probablility that a biased coin is heads. Let random variable N be the number of tosses up to and including the kth heads.

(a) What is the probability distribution obeyed by N? That is, what is $\Pr(N = \ell)$?

(b) Compute $E(N)$.

Problem 2. Let X_1, X_2 be Poisson random variables with parameters λ_1 and λ_2 respectively. Show that random variable $X = X_1 + X_2$ also obeys a Poisson distribution. What is its parameter?

Problem 3. Suppose a person walks along a straight line at a point 0. He walks at a uniform speed of 1 step per minute. But at the end of each minute he is likely to reverse his direction with probability 1/2. State your assumptions and bound the probability that he will be at a spot 100 steps away from 0 at the end of t minutes.

Problem 4. If m identical balls are thrown randomly into n identical bins, what is the expected number of empty bins?

Problem 5. What is the expected number of runs of heads in n (fair) coin flips? (Hint: Define an indicator random variable for the event that the ith flip ends a run of heads, and use linearity of expectation.)

Problem 6. How many $n \times n$ (0-1)-matrices have no all-0 columns?

Problem 7. In 400 rolls of a fair six-sided die, are you more likely to get 220 even numbers or to get 85 sixes? Justify your answer.

Problem 8. Let G be an undirected graph with $2n$ vertices. Argue carefully that if G contains no triangles, then G contains at most n^2 edges, and that this bound is tight.

Problem 9. Let X be a set of size n, and let $S_1, S_2, \cdots, S_b$ be a collection of (not necessarily distinct) subsets of X, all of size l. Suppose there exist integers t and λ such that if Y is any subset of X of size t, then $Y \subseteq S_i$ for exactly λ of the sets S_i. (Of course, $1 \leq t \leq l$ and $1 \leq \lambda \leq b$.) Suppose that R is a subset of X of size k, for some k, where $1 \leq k \leq t$. Argue that R is a subset of exactly $\lambda \binom{n-k}{t-k} / \binom{l-k}{t-k}$ of the subsets S_i.

Problem 10. Let $T(n) = T(2n/3) + T(n/4) + \sqrt{n}$ and $T(n)$ constant for small n. Is $T(n) = O(\sqrt{n})$?

(As is often the case, it is helpful to expand the recurrence by drawing the tree.)

Problem 11. Let X and Y be the outcomes of two independent rolls of a six-sided die. (An unbiased die with faces labelled $1, 2, \ldots, 6$ as usual). Let $Z = X \cdot 2^Y$. Find $E(Z)$ and $\mathrm{Var}(Z)$.