Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Mauricio Karchmer and Ron Rivest

Handout 28
May 5, 1995

Problem Set 9 Solutions

Problem 9-1. Suppose A, B, C are independent events. Show that any of the combinations

$$A^c, B, C;\ A, B^c, C; \cdots; A^c, B^c, C; \cdots; A^c, B^c, C^c$$

are also independent. Furthermore, show that A and $B \cup C$ are independent.

First note that there are a lot of symmetries that we can take advantage of here. The following lemmas are sufficient to prove the result

1. A and B^c are independent

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$

2. A^c and B^c are independent

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

3. $P(A^c \cap B \cap C) = P(A^c)P(B)P(C)$

$$\begin{aligned} P(A^c \cap B \cap C) &= P(A^c \cap (B \cap C)) \\ &= P(A^c \mid B \cap C) \cdot P(B \cap C) \\ &= (1 - P(A \mid B \cap C)) \cdot P(B \cap C) \\ &= (1 - P(A)) \cdot P(B) \cdot P(C) \\ &= P(A^c) \cdot P(B) \cdot P(C) \end{aligned}$$

Here the second equality follows from the multiplication theorem. The third is the property of conditional probability and the fourth is because A, B, C are independent. Therefore, A^c, B, C are independent. Given that A, B, C are symmetric, it is easy to see that A, B^c, C or A, B, C^c are also independent.

4. $P(A \cap B^c \cap C^c) = P(A)P(B^c)P(C^c)$

$$\begin{aligned}
P(A \cap B^c \cap C^c) &= 1 - P(A^c \cup B \cup C) \\
&= 1 - P(A^c) - P(B) - P(C) + P(A^c)P(B) + P(A^c)P(C) \\
&\quad + P(B)P(C) - P(A^c)P(B)P(C) \\
&= P(A) - P(B) - P(C) + (1 - P(A))P(B) + (1 - P(A))P(C) \\
&\quad + P(B)P(C) - (1 - P(A))P(B)P(C) \\
&= P(A) - P(A)P(B) - P(A)P(C) + P(A)P(B)P(C) \\
&= P(A)(1 - P(B))(1 - P(C)) \\
&= P(A)P(B^c)P(C^c)
\end{aligned}$$

5. $P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c)$

$$\begin{aligned}
P(A^c \cap B^c \cap C^c) &= 1 - P(A \cup B \cup C) \\
&= 1 - P(A) - P(B) - P(C) + P(A)P(B) + P(A)P(C) \\
&\quad + P(B)P(C) - P(A)P(B)P(C) \\
&= (1 - P(A))(1 - P(B))(1 - P(C)) \\
&= P(A^c)P(B^c)P(C^c)
\end{aligned}$$

Finally,

$$\begin{aligned}
P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) \\
&= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \\
&= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A \cap (B \cap C)) \\
&= P(A) \cdot (P(B) + P(C)) - P(A) \cdot P(B \cap C \mid A) \\
&= P(A) \cdot (P(B) + P(C) - P(B \cap C)) \\
&= P(A) \cdot P(B \cup C)
\end{aligned}$$

Therefore, A and $B \cup C$ are also independent.

Problem 9-2. Chuck-A-Luck is played in many casinos. One version of it is as follows. There is a cage with three dice in it. First the player bets \$1 on a number between 1 and 6. The three dice are then rolled. If i dice ($i = 1, 2$ or 3) show his number, he wins \$$i$ plus getting his initial \$1 bet back; if none of the dice shows the number, he loses his bet.

Define the expected value of some random variable X to be $E(X) = \sum_{\text{all x}} x \cdot \text{Prob}\{X = x\}$.

For a player starting with \$1, what is his expected wealth after one play?

Let random variable Y be the number of dice showing the chosen number, and X be his wealth after one play. Then $X = 0$ if $Y = 0$ and $X = i + 1$, if $Y = i$, for $i = 1, 2, 3$.

$P(Y = i) = \binom{3}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{3-i}$, $i = 0, 1, 2, 3$. Therefore,

$$\begin{aligned}
E(X) &= 0 \cdot P(Y = 0) + \sum_{i=1}^{3}(i + 1) \cdot P(Y = i) \\
&= \sum_{i=1}^{3}(i + 1) \cdot \binom{3}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{3-i} \\
&= \frac{1}{216}(2 \cdot 75 + 3 \cdot 15 + 4 \cdot 1) \\
&= 199/216 = .921
\end{aligned}$$

Problem 9-3. [10 points]

You are given a biased coin and are told that it could either be biased towards heads (with $P[\mathrm{H}] = \frac{2}{3}$) or biased towards tails (with $P[\mathrm{T}] = \frac{2}{3}$). Your best knowledge is that it is equally likely to be either kind of coin. So you flip the coin 12 times and you get 7 heads and 5 tails. What is your best guess about the type of coin and what is your estimate of the probability that you are right?

Define event A = the coin being biased towards heads (with $P[\mathrm{H}] = \frac{2}{3}$)
and event B = getting 7 heads out of 12 coin flips. Then

$$P(A) = P(A^c) = 0.5$$

$$P(B \mid A) = \binom{12}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^5 = 2^7 c$$

$$P(B \mid A^c) = \binom{12}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^5 = 2^5 c$$

where $c = \binom{12}{7}\left(\frac{1}{3}\right)^{12}$. And

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{2^7 c \times .5}{2^7 c \times .5 + 2^5 c \times .5} = .8$$

So with 80% probability, you are given a coin biased towards heads.

Problem 9-4. [10 points]

Once upon a time, a soldier was sentenced to death. To show his kindness, the king decided to give him another chance. He gave the soldier 50 black balls and 50 white balls and 2 pots and told him to divide the balls between the two pots in whatever way he liked. Then the king would come back and randomly take away one pot and mix the balls in the other. The soldier would then pick a ball from the remaining pot and live if and only if it was white. What strategy should the soldier take? Prove that it is the best.

He should put 1 white ball in one pot and all the other balls in the other. That gives him a $\frac{1}{2}(1+\frac{49}{99}) = .747$ chance of surviving. To prove that this strategy is optimal, suppose he had decided to put total z balls into the first pot. First note that $z > 0$, since if $z = 0$, then he puts all balls into one pot. The probability of surviving will be at most $\frac{1}{2}$. It would be $\frac{1}{2}$ if the king, on seeing an empty pot just makes him choose from the full pot and $\frac{1}{4}$ if he is executed if the king chooses the empty pot (the rules weren't entirely clear on what happens with an empty pot). This is clearly dominated by the strategy we just described. Also, without loss of generality, we can assume $z \leq 50$.(since at one pot has at most 50 balls) Suppose that the z balls consist of x white and y black ($0 \leq x, y \leq z$). For any chosen z, the soldier needs to find the x to maximize the probability of surviving. If $F_z(x)$ is the probability of surviving when we put x white balls into the first pot (containing z balls in all):

$$F_z(x) = \frac{1}{2} \cdot \frac{x}{z} + \frac{1}{2} \cdot \frac{50-x}{100-z}$$

Taking derivatives with respect to x, we get

$$F_z'(x) = \frac{1}{2} \cdot \left(\frac{1}{z} - \frac{1}{100-z}\right) = \frac{100-2z}{2z(100-z)} \geq 0$$

since we have assumed that $z \leq 50$.

Therefore, to maximize $F_z(x)$, we need to choose x as large as possible. So, taking $x = z$ would do the trick. Given that he always puts all white balls in the first pot, the soldier needs to find the $z \leq 50$ to maximize the probability of surviving:

$$S(z) = \frac{1}{2} \cdot \left(\frac{z}{z} + \frac{50-z}{100-z}\right) = 1 - \frac{50}{200-2z}$$

which clearly decreases with z. So the best strategy is to take z as small as possible. Therefore, $z = 1$ would maximize his chance of surviving.

Problem 9-5. [10 points]

Consider in a country where people favor boys over girls and each family will keep having babies until they have a boy. (And then they stop having kids.) Assume that genetically boys and girls are equally likely to be born, and that the death rate is the same for both sexes. Will there be a bias in that country's population in the long run? Explain your answer.

No, since each new baby has equal probability of being a boy or a girl. Consider any given time period with n births. The expected number of boys should be $\frac{n}{2}$.

Problem 9-6. [10 points] Mr. James W. Pitman received a letter:
" $\cdots$ to inform you that one of the following three people has won a New Datsun Sentra: Collin Andrus; James W. Pitman; Larry Abbott. $\cdots$"
" $\cdots$ you will receive one of the following for sure:
(1) RCA Color TV; (2) 5 FT. Grandfather Clock; (3) Datsun Nissan sentra.
To claim your gift, call $\cdots$"
According to the small print on the back of the letter:
" $\cdots$ the retail values and chance of receiving each gift are No. 1– $1/10,000$ (\$329.95), No.2– $9998/10,000$ (\$249.95), No.3– $1/10,000$ (\$5,995.00). $\cdots$ "
Let us assume that this is an honestly conducted sweepstake, and each of the three individuals named above had originally a 1 in 10,000 chance of winning the new Datsun. Now that Mr. Pitman knows that the winner is one of three individuals (and he is on the list), the rules of conditional probability imply that he has a $\frac{1}{3}$ chance of winning the Datsun. True or False? Explain!

False. The people that run the sweepstake do the following: they pick the winner and then pair up the rest of the people. They then send the same letter to each triple that consists of the winner and two other people. So does everyone who receives a letter have a $\frac{1}{3}$ chance of winning? Of course not. Lets formalize this:

$$\text{Prob}\{\text{winning} \mid \text{receive letter}\} = \frac{\text{Prob}\{\text{winning and receive letter}\}}{\text{Prob}\{\text{receive letter}\}} = \text{Prob}\{\text{winning}\}$$

since the probability of receiving a letter is 1.