Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 27
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Problem Set 10

Due: In recitation, May 12.

Problem 10-1. **(a)** Let X and Y be two random variables defined on the same probability space. Show that:

$$E(X) = \sum_k \Pr(Y = k) E(X \mid Y = k)$$

(b) Consider the following random experiment. We have n (fair) dice in a bin. We shake the bin and tip it over. Some of the dice roll out (it is equally likely that any number from 1 to n will roll out). We then sum up all of the values on the top faces of the dice that roll out. What is the expected value of this sum?

Problem 10-2. Suppose we toss a fair coin n times. We define the random variables X_i ($i = 1, \ldots, n$) by:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th toss is heads} \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \sum_{i=1}^n X_i$. Compute the following (where $a, b \in \{0, 1\}$ and $0 \leq k \leq n$):

(a) $\Pr(X_i = a \ \&\ X_j = b \mid X = k)$

(b) $\Pr(X = k \mid X_i = a \ \&\ X_j = b)$ (where $a, b \in \{0, 1\}$ and $0 \leq k \leq n$)

Problem 10-3. You are given a wheel (like a roulette wheel) which contains the numbers from 1 to $2n$ arranged in some way. The wheel has a spinner and on each spin you get the two numbers at the opposite ends of the spinner. How would you arrange the numbers on the wheel to maximize the expected value of:

(a) The sum of the numbers chosen?

(b) The product of the numbers chosen?

Problem 10-4. Let S be a sample space. Say random variable $X : S \to \mathbb{N}$ has distribution

$$X = \begin{cases} 2 & \text{with probability } 1/4, \\ 3 & \text{with probability } 1/4, \\ 5 & \text{with probability } 1/2; \end{cases}$$

and random variable $Y : S \to \mathbb{N}$ has distribution

$$Y = \begin{cases} 1 & \text{with probability } 2/3, \\ 4 & \text{with probability } 1/3; \end{cases}$$

where X and Y are independent.

Let random variable $Z : S \to \mathbb{N}$ be given by $Z = \max(X, Y)$.

(a) What is the expected value of Z?

(b) What is the variance of Z?

Problem 10-5. We are given a random vector of n distinct numbers. We then determine the maximum of these numbers by the following procedure:

Pick the first number. Call it the "current maximum". Go through the rest of the vector (in order) and each time we find a number x that exceeds our "current maximum", we update the "current maximum" with x.

What is the expected number of times we update the "current maximum"?

Problem 10-6. **(a)** Show that if random variables X and Y are independent then $\mathrm{Cov}(X, Y) = 0$

(b) Show that having $\mathrm{Cov}(X, Y) = 0$ does not necessarily mean that X and Y are independent.

(c) Let $X_1, \ldots, X_n$ be random variables. Prove that

$$\mathrm{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j)$$