Massachusetts Institute of Technology | Handout 26
6.042J/18.062J: Mathematics for Computer Science | April 28, 1995
Professors Mauricio Karchmer and Ron Rivest

Quiz 2 Solutions

Problem 2-1. Prove that if

$$\Lambda \cup \{\psi\} \models \chi \text{ and } \Lambda \cup \{\phi\} \models \chi \text{ and } \Lambda \models \psi \vee \phi$$

then

$$\Lambda \models \chi$$

In Lecture notes 10 section 7, it is proved that the system is sound and complete. That is,

$$\Gamma \vdash A \text{ implies } \Gamma \models A; \text{ and } \Gamma \models A \text{ implies } \Gamma \vdash A.$$

1. $\Lambda \cup \{\psi\} \vdash \chi$ — completeness.
2. $\Lambda \cup \{\phi\} \vdash \chi$ — completeness.
3. $\Lambda \vdash \psi \vee \phi$ — completeness.
4. $\Lambda \vdash \chi$ — Rule 8 and step $1, 2, 3$.
5. $\Lambda \models \chi$ — step 4 and soundness.

Problem 2-2. Prove or disprove: $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $g(n)$ is nondecreasing, $g(1) > 1$ and $f(n) \geq 0$ for all sufficiently large n.

We will assume that $f(n) \geq 1$ in this problem. Given $f(n) = O(g(n))$, we know that there exist some constants c and n_0, such that $\forall n \geq n_0$, $f(n) \leq cg(n)$. Choose a constant $c_2 > \dfrac{\lg c}{\lg(g(n_0))} + 1$, so $\lg c < (c_2 - 1)\lg(g(n_0))$. Then we have $\forall n \geq n_0$,

$$\begin{aligned}
\lg(f(n)) &\leq \lg(cg(n)) \\
&= \lg c + \lg(g(n)) \\
&< (c_2 - 1)\lg(g(n_0)) + \lg(g(n)) \\
&< (c_2 - 1)\lg(g(n)) + \lg(g(n)) \\
&= c_2 \lg(g(n))
\end{aligned}$$

The first inequality is because $f(n) \leq cg(n)$ and $\lg(\cdot)$ is strictly increasing. The second to last inequality is because $n \leq n_0$, g(n) is nondecreasing and $\lg(\cdot)$ is strictly increasing. Therefore, $\lg(f(n)) = O(\lg(g(n)))$.

Problem 2-3. Solve the following recurrence:

$$f(1) = f(2) = 1$$

$$f(n) - f(n-1) + f(n-2) = 3^n$$

This is a nonhomogenous linear recurrence. First, we isolate and solve the homogeneous part $f(n) - 4(n-1) + 4(n-2) = 0$. We try the exponential solution $f(n) = \alpha^n$. Substituting into the recurrence we get $\alpha^n - 4\alpha^{n-1} + 4\alpha^{n-2} = 0$. This gives us the characteristic equation, $\alpha^2 - 4\alpha + 4 = 0$. The roots of this equation are $\alpha_1 = \alpha_2 = 2$. Hence, $f(n) = c_1 2^n + c_2 n 2^n$, for some constants c_1 and c_2. Second, we obtain a particular solution for the nonhomogenous part. We try a solution of the form $c3^n$. Substituting into the recurrence we get $c3^n - 4c3^{n-1} + 4c3^{n-2} = 3^n$, which gives us the constant $c = 9$. Now we know that the general solution has the form $f(n) = c_1 2^n + c_2 n 2^n + 9 \cdot 3^n$. We solve for c_1 and c_2 using the initial condition, $f(1) = c_1 \cdot 2 + c_2 \cdot 2 + 27 = 1$ and $f(2) = c_1 \cdot 4 + c_2 \cdot 8 + 81 = 1$. We get $c_1 = -6, c_2 = -7$. Therefore, $f(n) = -6 \cdot 2^n - 7 \cdot n2^n + 3^{n+2}$.

Problem 2-4. Give tight (Θ) bounds for $T(n)$. Assume that $T(n)$ is constant for $n \leq 2$.

(a) $T(n) = 7T(n/3) + n^2$ Iterating, we see that

$$\begin{aligned}
T(n) &= n^2 + 7\left(\left(\frac{n}{3}\right)^2 + 7\left(\left(\frac{n}{3^2}\right)^2 + 7\left(\left(\frac{n}{3^3}\right)^2 + \cdots\right)\right)\right) \\
&= n^2 + 7\left(\frac{n}{3}\right)^2 + 7^2\left(\frac{n}{3^2}\right)^2 + 7^3\left(\frac{n}{3^3}\right)^2 + \cdots \\
&= n^2\left(1 + \left(\frac{7}{3^2}\right) + \left(\frac{7}{3^2}\right)^2 + \left(\frac{7}{3^2}\right)^3 + \cdots\right) \\
&= \Theta(n^2).
\end{aligned}$$

since we have a decreasing geometric series.

(b) $T(n) = T(2n/3) + T(n/4) + n$

> Note you cannot solve this by combining the solution for the two recurrences $T(n) = T(2n/3) + n$ and $T(n) = T(n/4) + n$. For example, $T(n) = T(2n/3) + T(n/2) + n$ does not grow as $\Theta(n)$ even though $T(n) = T(2n/3) + n$ and $T(n) = T(n/2) + n$ do.

To see that the answer should be $O(n)$, iterate the recurrence. Drawing the tree helps:

At each level, the branches to the right give 1/4 of the preceding level, and the branches to the left give 2/3 of the preceding level, for a total of 11/12 of the preceding level. This gives

$$n + (11/12)n + (11/12)^2 n + \ldots = \Theta(n).$$

We can prove $T(n) = O(n)$ by induction. First we try to show that there exists constant c_1 such that $T(n) \le c_1 n$. Substituting our guess in the recurrence, we obtain

$$T(n) \le c_1 \frac{2}{3} n + c_1 \frac{1}{4} n + n = \frac{11}{12} c_1 n + n \le c_1 n$$

which is true for, say, $c_1 = 12$. Therefore, $T(n) = O(n)$. Clearly $T(n) \ge n$, so $T(n) = \Omega(n)$ also. Thus, $T(n) = \Theta(n)$.

(c) $T(n) = 4T(n/3) + n$

We may again iterate as in part **(a)** to see that

$$\begin{aligned}
T(n) &= n\left(1 + \left(\frac{4}{3}\right) + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{\log_3 n}\right) \\
&= \Theta\left(n\left(\frac{4}{3}\right)^{\log_3 n}\right) \\
&= \Theta\left(n\,(n)^{\log_3 4/3}\right) \\
&= \Theta\left(n^{1+\log_3 4/3}\right) \\
&= \Theta\left(n^{\log_3 4}\right).
\end{aligned}$$

Problem 2-5. **(a)** $S(n,k)\,k!$ is the number of ways to color n (distinguishable) balls using exactly k colors. First partition the balls into k (nonempty) classes (all the balls in each class will be colored the same). By definition, there are $S(n,k)$ ways to do this. Now assign colors to the classes. This can be done in $k!$ ways. (Recall that there is no simple closed-form for $S(n,k)$.)

If you like, you may think of different colors as bins. This is the problem of throwing n different balls into k different nonempty bins. We studied this problem in Lecture 17, where we noticed that the answer, denoted $T(n,k)$, equals $k!\,S(n,k)$ where $S(n,k)$, the Stirling number of the second kind, is the answer for *indistinguishable* bins. We proceeded to derive an expression for $T(n,k)$ using inclusion/exclusion.

(b) $S(n,k)\,x(x-1)\cdots(x-k+1)$ is the number ways to color the n balls using exactly k colors if $x > k$ colors are available.

This is the same as part **(a)** except that the number of ways to assign colors to the classes is $x(x-1)\cdots(x-k+1)$.

Alternatively, we may think of first choosing the k colors to use, and then coloring as in part **(a)**, which gives $\binom{x}{k}T(n,k) = \binom{x}{k}S(n,k)\,k!$, which is the same.

(c)

$$x^n = \sum_{k=1}^{n} S(n,k)\cdot x(x-1)\cdots(x-k+1).$$

Each side counts the number of ways to color n different balls using x colors.

This is clear for the LHS: there are x ways to color each of the n balls.

The RHS counts in an alternative way: First choose the number k of colors $(1 \le k \le n)$ actually to be used, then color the balls using exactly k colors, as done in part **(b)**.

Each side is a polynomial of degree n in x. Since the two sides agree for any positive integer x, in particular they agree on $n+1$ points and therefore are equal—this is a an identity.

Problem 2-6. How many permutations of ABCDEFG are there that don't contain "$\cdots$BEG$\cdots$" or "$\cdots$AD$\cdots$"?

Define $X = \{$permutations of ABCDEFG$\}$
$Y = \{$permutations of ABCDEFG that contain BEG$\}$
$Z = \{$permutations of ABCDEFG that contain AD$\}$

So, we need to find

$$|X| - |Y \cup Z| = |X| - |Y| - |Z| + |Y \cap Z|$$

$|X| = 7!$ is trivial. $|Y| = 5!$ since you can just consider BEG as one letter: the number of permutations of $\{$A, BEG, C, D, F$\}$ is $5!$. Similarly, $|Z| = 6!$ and $|Y \cap Z| = 4!$.
Answer: $7! - 5! - 6! + 4! = 4224$.