Massachusetts Institute of Technology | Handout 22
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Problem Set 8 Solutions

Problem 8-1. [5 points] Discover and prove a formula for

$$\sum_{k=0}^{n} k^3 \binom{n}{k}$$

We will do this analytically, much like $\sum_{k=0}^{n} k\binom{n}{k}$. We know from the binomial theorem that

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiating both sides gives us

$$n(x+1)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

and also

$$n(n-1)(x+1)^{n-2} = \sum_{k=2}^{n} k(k-1) \binom{n}{k} x^{k-2}$$

Substituting $x = 1$ and accounting for the missing terms (we started the summation at $k = 2$) gives

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

and

$$n(n-1)2^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k} - n - \sum_{k=0}^{n} k \binom{n}{k} + n$$

thus,

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1} = n(n+1)2^{n-2}$$

We now have all of the pieces we need to find the required sum. By differentiating once more we have

$$\begin{aligned} n(n-1)(n-2)(x+1)^{n-3} &= \sum_{k=3}^{n} k(k-1)(k-2) \binom{n}{k} x^{k-3} \\ &= \sum_{k=3}^{n} (k^3 - 3k^2 + 2k) x^{k-3} \end{aligned}$$

Substituting $x = 1$ once more and gathering up the missing terms we have

$$\begin{aligned} n(n-1)(n-2)2^{n-3} &= \sum_{k=0}^{n} k^3 \binom{n}{k} - n - 8\binom{n}{2} - 3\left[n(n+1)2^{n-2} - n - 4\binom{n}{2}\right] \\ &\quad +2\left[n2^{n-1} - n - 2\binom{n}{2}\right] \\ &= \sum_{k=0}^{n} k^3 \binom{n}{k} - 3\sum_{k=0}^{n} k^2 \binom{n}{k} + 2\sum_{k=0}^{n} k\binom{n}{k} \\ &= \sum_{k=0}^{n} k^3 \binom{n}{k} - 3n(n+1)2^{n-2} + n2^n \end{aligned}$$

Rearranging terms gives us the solution

$$\begin{aligned} \sum_{k=0}^{n} k^3 \binom{n}{k} &= n(n-1)(n-2)2^{n-3} + 3n(n+1)2^{n-2} - n2^n \\ &= 2^{n-3}(n(n-1)(n-2) + 6n(n+1) - 8n) \\ &= 2^{n-3}(n^3 + 3n^2) \\ &= n^2(n+3)2^{n-3} \end{aligned}$$

Problem 8-2. [5 points] How many m digit numbers are there such that each one starts with a significant digit (i.e. not 0) and all digits appear in each number?

Let $A(n, k)$ be the number of n digit numbers where each digit (we have k of them) appears at least once. We know from Lecture 16 that $A(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$. To count those that start with a significant digit we break them up into two cases:

1. The digit that it starts with appears again in the number. We have $A(m-1, k)$ different numbers for each significant digit
2. The digit does not appear again. We have $A(m-1, k-1)$ different numbers for each significant digit

Thus the total count equals $(k-1)(A(m-1,k) + A(m-1,k-1)) =$

$$(k-1)\left[\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^{m-1} + \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j}(k-1-j)^{m-1}\right]$$

Problem 8-3. [10 points] Prove that the Bell numbers satisfy the recurrence

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Proof. We know that B_{n+1} counts the number of partitions of $\{1,\ldots,n+1\}$. Let's pick an element of the set and call it x. We now count all of the partitions in which the class containing x has j elements. There are $\binom{n}{j-1}B_{n+1-j}$ such partitions. Why? There are $\binom{n}{j-1}$ ways of choosing the remaining elements in x's class and B_{n+1-j} ways of partitioning the rest of the elements.

Thus the total number of partitions of $\{1,\ldots,n+1\}$ is the sum over all j of the above expression

$$\begin{aligned} B_{n+1} &= \sum_{j=1}^{n+1}\binom{n}{j-1}B_{n+1-j} \\ &= \sum_{l=0}^{n}\binom{n}{l}B_{n-l} \\ &= \sum_{l=0}^{n}\binom{n}{n-l}B_{n-l} \\ &= \sum_{k=0}^{n}\binom{n}{k}B_k \end{aligned}$$

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Problem 8-4. [10 points] Calculate how many ways 3 distinct numbers can be selected from $\{1,2,3,\ldots,150\}$ so that their sum is

(a) even. There are 75 even and 75 odd numbers in the given interval. To choose 3 distinct numbers so that their sum is even we either choose

1. 3 even numbers. We can do this in $\binom{75}{3}$ ways.
2. 2 odd numbers and 1 even number. We can do this in $\binom{75}{2} * 75$ ways.

Thus the total number is $75 * \binom{75}{2} + \binom{75}{3} = 275650$.

(b) odd. The sum of the three numbers is either even or odd. Thus the number of different ways we can choose three distinct numbers so that their sum is odd is $\binom{150}{3}-$(answer from part(a)). This equals 275650.

(c) a multiple of 3. In the interval we are given there are 50 numbers whose remainder when divided by 3 is 0, 50 whose remainder is 1 and 50 whose remainder is 2. For the sum of the 3 numbers to be a multiple either

1. All 3 numbers have the same remainder when divided by 3. We can choose these in $3\binom{50}{3}$ ways.
2. Their remainders as all different. We can choose these in 50^3 ways.

Thus the total number of ways is $3\binom{50}{3} + 50^3 = 183800$.

Problem 8-5. [20 points] In this problem we will prove that the number of labeled trees on n vertices is n^{n-2}.

(a) Prove the following generalization of Pascal's triangle

$$\binom{m}{m_1 \dots m_k} = \sum_{i=1}^{k} \binom{m-1}{m_1 \dots m_i - 1 \dots m_k}$$

Proof.

We know from the multinomial theorem that the coefficient of $x_1^{m_1} \dots x_k^{m_k}$ in $(x_1 \dots + x_k)^m$ is $\binom{m}{m_1 \dots m_k}$. Another way of computing this coefficient is by noting that

$$(x_1 + \dots + x_k)^m = (x_1 + \dots + x_k)^{m-1}(x_1 + \dots + x_k)$$

We can thus get $x_1^{m_1} \dots x_k^{m_k}$ by multiplying $x_1^{m_1} \dots x_i^{m_i - 1} \dots x_k$ by x_i for $i = 1, \dots, k$. Thus the required coefficient is

$$\sum_{i=1}^{k} \binom{m-1}{m_1 \dots m_i - 1 \dots m_k}$$

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(b) Let $t(n; d_1, \dots, d_n)$ be the number of trees on n vertices such that the degree of vertex i is d_i. Show that

$$t(n; d_1, \dots, d_{n-1}, 1) = \sum_{i=1}^{n-1} t(n-1; d_1, \dots, d_i - 1, \dots, d_{n-1})$$

We know that vertex n has degree 1 so it's a leaf. It must be connected to some vertex, call it i. We will now count how many trees have vertex n connected to vertex i. If we remove vertex n we have a tree on $n-1$ vertices where the degree sequence is almost the same – vertex i has degree $d_i - 1$. Thus we have $t(n-1; d_1, \dots, d_i - 1, \dots, d_{n-1})$ such trees. So to find the number of trees on n vertices we just sum the expression above from $i = 1$ to $n - 1$

$$\sum_{i=1}^{n-1} t(n-1; d_1, \dots, d_i - 1, \dots, d_{n-1})$$

(c) Show that the number of trees on n vertices such that the degree of vertex i is d_i is

$$\binom{n-2}{(d_1 - 1)(d_2 - 1) \dots (d_n - 1)}$$

(Hint: Use parts (a) and (b) and induction on n)

We will do this by induction on n.

Base Case: $n = 2$. Both are equal to 1.

Inductive step: We know that

$$t(n; d_1, \ldots, 1) = \sum_{i=1}^{n-1} t(n-1; d_1, \ldots, d_i - 1, \ldots, d_{n-1})$$

We must now find a formula for $t(n; d_1, \ldots, d_n)$. Without loss of generality we can assume that $d_n = 1$. (t is independent of the order of the degree sequence and there must be some leaf). Thus,

$$\begin{aligned}
t(n; d_1, \ldots, d_n) &= \sum_{i=1}^{n-1} t(n-1; d_1, \ldots, d_i - 1, \ldots, d_{n-1}) \\
&= \sum_{i=1}^{n-1} \binom{n-3}{(d_1-1)\ldots(d_i-1-1)\ldots(d_{n-1}-1)} && \text{(Inductive hypothesis)} \\
&= \binom{n-2}{(d_1-1)\ldots(d_{n-1}-1)} && \text{(part (a))} \\
&= \binom{n-2}{(d_1-1)\ldots(d_n-1)} && (d_n - 1 = 0)
\end{aligned}$$

(d) Now prove that the number of labeled trees on n vertices is n^{n-2}.

Proof.

To find the total number of trees we just sum the expression above over all degree sequences. If we let $k_i = d_i - 1$ the sum is

$$\sum_{\substack{k_1, \ldots, k_n \\ k_1 + \ldots + k_n = n-2}} \binom{n-2}{k_1 \ldots k_n} = n^{n-2} \quad \text{(from the multinomial theorem)}$$

Note that since we have a tree the sum of the degrees is $2(n-1) = 2n-2$ and so $\sum_{i=1}^{n} k_i = n - 2$.

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