Massachusetts Institute of Technology
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Problem Set 7 Solutions

Problem 7-1. Solve the following recurrences:

(a) $f(n) = 2f(n-1) - 2f(n-2);\ f(0) = 0,\ f(1) = 1.$
This is a homogenous linear recurrence. So we try the exponential solution $f(n) = \alpha^n$. Substituting into the recurrence we get $\alpha^n = 2\alpha^{n-1} - 2\alpha^{n-2}$. This gives us the characteristic equation, $\alpha^2 - 2\alpha + 2 = 0$. The roots of this equation are $\alpha_1 = 1+i$ and $\alpha_2 = 1-i$. Hence, $f(n) = c_1(1+i)^n + c_2(1-i)^n$, for some constants c_1 and c_2. We solve for c_1, c_2 using the initial conditions: $f(0) = c_1 + c_2 = 0$ and $f(1) = c_1(1+i) + c_2(1-i) = 1$. We get $c_1 = -\frac{i}{2}$ and $c_2 = \frac{i}{2}$. Therefore, $f(n) = -\frac{i}{2}(1+i)^n + \frac{i}{2}(1-i)^n$.

(b) $f(n) = 7f(n-1) - 16f(n-2) + 12f(n-3);\ f(0) = 0,\ f(1) = 1,\ f(2) = 2.$
This is a homogenous linear recurrence. So we try the exponential solution $f(n) = \alpha^n$. Substituting into the recurrence we get $\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$. This gives us the characteristic equation, $\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$. $\alpha^3 - 7\alpha^2 + 16\alpha - 12 = (\alpha-2)^2(\alpha-3)$. The roots of this equation are $\alpha_1 = 2$, $\alpha_2 = 2$ and $\alpha_3 = 3$. The root 2 has multiplicity two. As shown in class, $2^n, n2^n$ and 3^n are solutions to the recurrence. Hence $f(n) = c_1 2^n + c_2 n 2^n + c_3 3^n$, for some constants c_1, c_2 and c_3 . We solve for c_1, c_2 and c_3 using the initial conditions: $f(0) = c_1 + c_3 = 0$, $f(1) = 2c_1 + 2c_2 + 3c_3 = 1$ and $f(2) = 4c_1 + 8c_2 + 9c_3$. We get $c_1 = 2, c_2 = \frac{3}{2}, c_3 = -2$. Therefore, $f(n) = 2^{n+1} + \frac{3}{2}n2^n - 2 \cdot 3^n$.

(c) $f(n) = 2f(n-1) + n;\ f(0) = 1.$
This is a nonhomogenous linear recurrence. First, we isolate and solve the homogeneous part $f(n) = 2f(n-1)$. We try the exponential solution and find that $\alpha = 2$ is the only root of the characteristic equation. Hence $f(n) = c2^n$, for some constant c. Second, we obtain a particular solution for the nonhomogenous part. Since the nonhomogenous part is a polynomial of degree 1 we try a general degree 1 polynomial $an + b$ as the particular solution. Substituting into the recurrence we get $an + b = 2(a(n-1) + b) + n$, or $(a+1)n - (2a-b) = 0$. Solving for a and b we find that, $a = -1$ and $b = -2$. Adding the solution to the homogenous part with the particular solution we find that the general solution is $f(n) = c2^n - n - 2$. We solve for c using the initial condition, $f(0) = c - 2 = 1$. We get $c = 3$. Therefore, $f(n) = 3 \cdot 2^n - n - 2$.

Problem 7-2. Prove that the Vandermonde matrix $\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \cdots & \alpha_d^1 \\ \vdots & & \vdots & \ddots & \\ \alpha_1^{d-1} & \alpha_2^{d-1} & \alpha_3^{d-1} & \cdots & \alpha_d^{d-1} \end{bmatrix}$ has determinant $\prod_{0<j<k\le d} (\alpha_k - \alpha_j)$.

Proof. The proof is by induction on d.

Base case. $d = 2$ is trivial.

Inductive step. Assume the inductive hypothesis that the Vandermonde matrix of size $d-1$ has determinant $\prod_{0<j<k\le d-1}(\alpha_k - \alpha_j)$, for $d \ge 3$. From linear algebra, we know that we can subtract a multiple of one row from another row without changing the value of the determinant. Consider the Vandermonde matrix of size d and let D be the value of its determinant. Multiplying the ith row by α_1 and subtracting it from the $(i+1)$th row for $i = d-1, d-2, \cdots, 1$, we get that:

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1^1 - \alpha_1^1 & \alpha_2^1 - \alpha_1^1 & \alpha_3^1 - \alpha_1^1 & \cdots & \alpha_d^1 - \alpha_1^1 \\ \vdots & & \vdots & \ddots & \\ \alpha_1^i - \alpha_1^i & \alpha_2^{i-1}(\alpha_2 - \alpha_1) & \alpha_3^{i-1}(\alpha_3 - \alpha_1) & \cdots & \alpha_d^{i-1}(\alpha_d - \alpha_1) \\ \vdots & & \vdots & \ddots & \\ \alpha_1^{d-1} - \alpha_1^{d-1} & \alpha_2^{d-2}(\alpha_2 - \alpha_1) & \alpha_3^{d-2}(\alpha_3 - \alpha_1) & \cdots & \alpha_d^{d-2}(\alpha_d - \alpha_1) \end{vmatrix} \\
&= \prod_{1<k\le d} (\alpha_k - \alpha_1) \begin{vmatrix} 1 & 1/(\alpha_2 - \alpha_1) & 1/(\alpha_3 - \alpha_1) & \cdots & 1/(\alpha_d - \alpha_1) \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & & \vdots & \ddots & \\ 0 & \alpha_2^{i-1} & \alpha_3^{i-1} & \cdots & \alpha_d^{i-1} \\ \vdots & & \vdots & \ddots & \\ 0 & \alpha_2^{d-2} & \alpha_3^{d-2} & \cdots & \alpha_d^{d-2} \end{vmatrix}
\end{aligned}
$$

$$= \prod_{1<k\le d}(\alpha_k-\alpha_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ & \vdots & \ddots & \\ \alpha_2^{i-1} & \alpha_3^{i-1} & \cdots & \alpha_d^{i-1} \\ & \vdots & \ddots & \\ \alpha_2^{d-2} & \alpha_3^{d-2} & \cdots & \alpha_d^{d-2} \end{vmatrix}$$

$$= \left(\prod_{1<k\le d}(\alpha_k-\alpha_1)\right)\times\left(\prod_{1<j<k\le d}(\alpha_k-\alpha_j)\right) \text{ by inductive hypothesis.}$$

$$= \prod_{0<j<k\le d}(\alpha_k-\alpha_j)$$

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Problem 7-3. **(a)** How many ways can $2n$ people be divided into n pairs?

Answer: $(2n)!/n!2^n$.

Consider all the ways of ordering the $2n$ people in a line. There are $(2n)!$ ways of doing this. Now group them into pairs based on the linear order, i.e. group the first and second persons, group the third and fourth persons,..., group the $(2n-1)$th and $2n$th persons. Thus each linear order gives us a unique grouping into pairs. But observe that two different linear orders could give rise to the same grouping into pairs. So let us now count the number of linear orders that give rise to the same grouping into pairs. Given a grouping into pairs, to generate a linear order we need to order each pair (there are 2^n ways to do this) and then order the n pairs (there are $n!$ ways to do this). Hence $n!2^n$ linear orders give the same grouping into pairs. Thus the total number of ways $2n$ people can be divided into n pairs is $(2n)!/n!2^n$.

(b) How many ways can you choose n out of $2n$ objects, given that n of the $2n$ objects are identical?

The answer is 2^n, since you can pick any subset from the n nonidentical objects, and make up the rest with the identical ones. And the number of subsets of n different objects is 2^n.

Problem 7-4. A pizza house is having a promotional sale. Their commercial reads: "... buy 2 large pizzas at regular price, get up to 11 toppings for each pizza absolutely free. That's 4194304 different ways to design your order!!!! ..."

(a) Show that $4194304 = \left(\sum_{k=0}^{11} \binom{11}{k}\right)^2$ is actually wrong and give the right answer for the number of ways that you can choose toppings for the two pizzas.

The number of ways to choose different toppings for one pizza is clearly $A \equiv \sum_{k=0}^{11} \binom{11}{k} = 2^{11} = 2048$. However, when they say there are A^2 different ways to choose toppings for two pizzas, they are making the mistake of counting the combination "one mushroom pizza, and one cheese pizza" as different from the combination "one cheese pizza, and one mushroom pizza."

The right solution is $A + \binom{A}{2} = \frac{1}{2}(A^2 + A) = 2098176$, where A is the number of ways to choose 2 pizzas with the same toppings, and $\binom{A}{2}$ is the number of ways to choose 2 different pizzas.

(b) Now assume that you can choose any number of the same topping as long as the total number of toppings is no more than 11. (That is, you can have 2 meatball, 4 ham and 5 sausage toppings if you are truly a meat lover.) How many different ways can you now choose toppings for your two pizzas?

For a single pizza, this is the "balls and bins" problem in disguised form. If k is the total number of toppings you want on a pizza then the number of ways of throwing k identical balls into 11 bins is identical to choosing a total of k toppings out a total of 11 different topping-possibilities.

Hence the number of ways to order one pizza is $B \equiv \sum_{k=0}^{11} \binom{10+k}{k} = \binom{22}{11} = 705432$. Using the same argument as in part (a), we see that the number of ways to order two pizzas is $B + \binom{B}{2} = \frac{B^2 + B}{2} = 248817506028$.