Problem Set 8

Due: In recitation, April 21.

Problem 8-1. [5 points] Discover and prove a formula for

$$\sum_{k=0}^{n} k^3 \binom{n}{k}$$

Problem 8-2. [5 points] How many m digit numbers are there such that each one starts with a significant digit (i.e. not 0) and all digits appear in each number?

Problem 8-3. [10 points] Prove that the Bell numbers satisfy the recurrence

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Problem 8-4. [10 points] Calculate how many ways 3 distinct numbers can be selected from $\{1, 2, 3, \ldots, 150\}$ so that their sum is

(a) even

(b) odd

(c) a multiple of 3

Problem 8-5. [20 points] In this problem we will prove that the number of labeled trees on n vertices is n^{n-2}.

(a) Prove the following generalization of Pascal's triangle

$$\binom{m}{m_1 \ldots m_k} = \sum_{i=1}^{k} \binom{m}{m_1 \ldots m_i - 1 \ldots m_k}$$

(b) Let $t(n; d_1, \ldots, d_n)$ be the number of trees on n vertices such that the degree of vertex i is d_i. Show that

$$t(n; d_1, \ldots, d_{n-1}, 1) = \sum_{i=1}^{n-1} t(n-1; d_1, \ldots, d_i - 1, \ldots, d_{n-1})$$

(c) Show that the number of trees on n vertices such that the degree of vertex i is d_i is

$$\binom{n-2}{(d_1-1)(d_2-1)\dots(d_n-1)}$$

(Hint: Use parts (a) and (b) and induction on n)

(d) Now prove that the number of labeled trees on n vertices is n^{n-2}.