Problem Set 6 Solutions

Problem 6-1. Give an example of two functions f and g that are incomparable using asymptotic notation. (i.e. $f(n)$ is not $O(g(n))$, $g(n)$ is not $O(f(n))$, $f(n)$ is not $\Omega(g(n))$, and so on)

Here's an example:

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n2^n & \text{if } n \text{ is odd} \end{cases} \quad g(n) = \begin{cases} n2^n & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

$f(n)$ can't be $O(g(n))$ since it would imply that $n2^n \le cn$ for n large and odd. Similarly $f(n)$ can't be $\Omega(g(n))$ and so on.

Problem 6-2. Rank the following functions by order of growth; that is find an arrangement $g_1, g_2, \ldots, g_{30}$ of the functions satisfying $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{29} = \Omega(g_{30})$. Partition your list into equivalence classes such that $f(n)$ and $g(n)$ are in the same class iff $f(n) = \Theta(g(n))$.

$\lg(\lg^* n)$	$2^{\lg^* n}$	$(\sqrt{2})^{\lg n}$	n^2	$n!$	$(\lg n)!$
$(3/2)^n$	n^3	$\lg^2 n$	$\lg(n!)$	2^{2^n}	$n^{(1/\lg n)}$
$\ln \ln n$	$\lg^* n$	$n2^n$	$n^{\lg \lg n}$	$\ln n$	1
$2^{\lg n}$	$(\lg n)^{\lg n}$	e^n	$4^{\lg n}$	$(n+1)!$	$\sqrt{\lg n}$
$\lg^*(\lg n)$	$2^{\sqrt{2 \lg n}}$	n	2^n	$n \lg n$	$2^{2^{n+1}}$

Here's a definition for $\lg^* n$ ("log star of n"). We first define the function $\lg^{(i)} n$.

$$\lg^{(i)} n = \begin{cases} n & \text{if } i = 0, \\ \lg(\lg^{(i-1)} n) & \text{if } i > 0 \text{ and } \lg^{(i-1)} n > 0, \\ \text{undefined} & \text{if } i > 0 \text{ and } \lg^{(i-1)} n \le 0 \text{ or } \lg^{(i-1)} n \text{ is undefined} \end{cases}$$

Now $\lg^* n = \min\{i \ge 0 : \lg^{(i)} n \le 1\}$. For example $\lg^* 2 = 1, \lg^* 4 = 2, \lg^* 16 = 3, \lg^* 65536 = 4, \lg^*(2^{65536}) = 5$.

The ranking is: ($f(n) \prec g(n)$ means $f(n) = \Omega(g(n))$ and $f(n) \sim g(n)$ means $f(n) = \Theta(g(n))$).

$$\begin{aligned}1 &\sim n^{1/\lg n} \prec \lg\lg^* n \prec \lg^* \lg n \sim \lg^* n \prec 2^{\lg^* n} \prec \ln\ln n \prec \sqrt{\lg n} \prec \ln n \prec \lg^2 n \\ &\prec 2^{\sqrt{2\lg n}} \prec \sqrt{2}^{\lg n} \prec n \sim 2^{\lg n} \prec n\lg n \sim \lg n! \prec n^2 \sim 4^{\lg n} \prec n^3 \\ &\prec (\lg n)! \prec \lg n^{\lg n} \sim n^{\lg\lg n} \prec (3/2)^n \prec 2^n \prec n2^n \prec e^n \prec n! \prec (n+1)! \prec 2^{2^n} \prec 2^{2^{n+1}}\end{aligned}$$

Problem 6-3. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
This is false. 1 is $O(n)$ but n isn't $O(1)$.

(b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$. This is false. Let $f(n) = 1$, $g(n) = n$. $1 + n$ isn't $\Theta(1)$.

(c) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$. This is false. 2^{n+1} is $O(2^n)$ but $2^{2^{n+1}}$ isn't $O(2^{2^n})$. Why?

(d) $f(n) = O((f(n))^2)$. This is false. Let $f(n) = 1/n$. There is no way that $1/n \le c/n^2$ for n large enough for any constant c.

(e) $f(n) + o(f(n)) = \Theta(f(n))$.
This is true. We first show that $f(n) + o(f(n)) = O(f(n))$. Let $g(n) = o(f(n))$. We know that there exists c such that for n large enough $g(n) \le cf(n)$. Thus for n large enough, $f(n) + g(n) \le (1 + c)f(n)$.

$f(n) + o(f(n))$ is $\Omega(f(n))$. Since $f(n)$ and $g(n)$ are asymptotically positive, $f(n) + g(n) \ge f(n)$ for n large.

Problem 6-4. Prove that:

$$\binom{n}{n/2} = \Theta(2^n/\sqrt{n})$$

Please be careful when you're manipulating the Θs. Remember that you have to prove both an upper and a lower bound.

We use Stirling's approximation here. Recall that

$$n! = \sqrt{2\pi n}(n/e)^n(1 + \Theta(1/n))$$

So we know that for n large,

$$\begin{aligned} n! &\le \sqrt{2\pi n}(n/e)^n(1+a) \\ &\ge \sqrt{2\pi n}(n/e)^n \end{aligned}$$

for some constant a.

We first note that $\binom{n}{n/2} = \frac{n!}{(n/2)!^2}$ and so if we want to upper bound $\binom{n}{n/2}$ it suffices to upper bound the numerator and lower bound the denominator. Thus for n large,

$$\begin{aligned}\binom{n}{n/2} &\leq \frac{\sqrt{2\pi n}(n/e)^n d}{\left(\sqrt{\frac{2\pi n}{2}}(\frac{n}{2e})^{n/2}\right)^2} \\ &\leq \frac{c\sqrt{n}(n/e)^n}{n(\frac{n}{2e})^n}\text{for some constant } c \\ &\leq \frac{c2^n}{\sqrt{n}}\end{aligned}$$

Thus $\binom{n}{(n/2)} = O(2^n/\sqrt{n})$. To show that $\binom{n}{(n/2)} = \Omega(2^n/\sqrt{n})$ we basically do the same thing. We lower bound the numerator and upper bound the denominator. The first step is thus:

$$\binom{n}{n/2} \geq \frac{\sqrt{2\pi n}(n/e)^n}{\left(\sqrt{\frac{2\pi n}{2}}(\frac{n}{2e})^{n/2}d\right)^2}$$

and the other steps are similar.

Problem 6-5. Give asymptotic upper and lower bounds for $T(n)$ in the following recurrence.

$$T(n) = 2T(n/2) + n^3.$$

We use the master method here (see Handout 16). Note that $a = b = 2$ and so $\log_b a = 1$. Note also that $f(n) = n^3$ is $\Omega(n^{1+1})$ $(\epsilon = 1)$ and $2 * (n/2)^3 = \frac{1}{4}n^3 \leq \frac{1}{2}n^3$ $(c = 1/2)$. Case 3 thus holds. Therefore $T(n) = \Theta(n^3)$.